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An Inverse Initial Problem for a Fourth Order Time-Fractional Space Degenerate Partial Differential Equation

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Annotation: In this paper, we investigate inverse initial problem for a fourth-order differential equation that becomes degenerate along the boundary of a rectangular domain. By applying the method of separation of variables, we reduce the problem to a spectral problem for an ordinary differential equation. We then construct the Green's function for this spectral problem, which allows us to transform it into a Fredholm integral equation of the second kind with a symmetric kernel. Using the theory of integral equations with symmetric kernels, we establish the existence and some properties of the eigenvalues and eigenfunctions of the spectral problem. The solution to the original problem is expressed as a Fourier series expansion in terms of these eigenfunctions, and we prove the uniform convergence of this series.

Keywords: degenerate equation, an inverse initial problem, spectral method, Green's function, integral equation with symmetric kernel.

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1. INTRODUCTION

With the growing importance of science and engineering, inverse problems (IPs) have become a fundamental aspect of these disciplines. One prominent example is the heat equation, which plays a vital role in numerous scientific applications such as melting, freezing, manufacturing, microwave heating, medicine, biology, seismology, fluid flow in porous media, and seawater desalination, among others. These applications are typically described by partial differential equations (PDEs). When all input parameters-such as initial and boundary conditions, domain geometry, coefficients, and external forces-are known, it is possible to solve the PDEs and predict the system's behavior [1].

However, when some of this critical information is unavailable or unknown, predicting the system's behavior becomes challenging. In such cases, measurable outputs from experiments can be combined with the known inputs to recover the missing data. This process defines an inverse problem. Generally, IPs are considered ill-posed, meaning that small errors or variations in input data can lead to large deviations in the resulting solutions.

When dealing with inverse problems, the presence of additional unknown terms often necessitates the imposition of extra conditions-such as nonlocal integral conditions-to serve as overdetermination constraints. Foundational contributions to the theory and study of inverse problems were made by researchers like Ivanov [2], Lavrentiev et al. [3], and Tikhonov [4].

Only a limited number of studies have explored inverse problems involving higher-order partial differential equations (PDEs) with supplementary measurement data. For instance, works such as [5]-[7] focused on recovering time-dependent potential coefficients in fourth-order pseudo-parabolic equations. Similarly, in [8], [9], the authors examined inverse problems aimed at identifying unknown components in fourth- and sixth-order PDEs. Yuldashev [10] contributed by proving the unique solvability of inverse problems related to fourth-order PDEs.

Abbasova et al. [11] investigated the solvability of an inverse problem (IP) related to the linear equation of motion for a homogeneous elastic beam and established the existence and uniqueness of its classical solution. In their study, the time-dependent potential coefficient and temperature were numerically determined for the first time for a higher-order partial differential equation (PDE) using a nonlocal integral condition. This work introduced a completely new inverse problem that had not been previously examined in the literature. To solve the direct problem, the authors employed a quartic B-spline (QBS) collocation method for discretization. The inverse problem was approached by minimizing a least-squares objective functional to obtain a quasi-solution. The uniqueness of the potential coefficient had already been proven by the authors in [11] using the contraction mapping principle. A fundamental contribution of the work was the proposal and implementation of a regularization algorithm for solving the parameter identification problem. Due to the ill-posed nature of the inverse problem, the Tikhonov regularization method was utilized to achieve stable numerical results.

In the present paper, we study an inverse initial problem for a fourth-order partial differential equation with degeneracy.

2. PRELIMINARY MATERIALS

2.1. Generalized Mittag-Leffler function. In this subsection, we will recall the definition of the generalized Mittag-Leffler function and give its necessary properties which we will use in our investigation. The generalized Mittag-Lefler function is

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},\ z\in\square\ .$$

Lemma 2.1. Suppose that $\alpha\in(0,2)$, β is an arbitrary real number and, $\pi\alpha/2<\mu<\min\{\pi,\pi\alpha\}$. Then there exits a positive constant C' such that the equality holds:

$$\left|E_{\alpha,\beta}(z)\right|\le\frac{C'}{1+|z|},$$

$$\text{for all } \mu\le\left|\arg(z)\right|\le\pi \text{ and } |z|\ge 0\,.$$

Note that in the following estimate for the Mittag-Leffler function is proved, when $0<\alpha<1$ (not true for $\alpha\ge 1$)

$$\frac{1}{1+\Gamma(1-\alpha)z}\le E_{\alpha,1}(-z)\le\frac{1}{1+\Gamma(1+\alpha)^{-1}z},\quad z>0\,.$$

Thus, it follows that

$$0 < E_{\alpha,1}(-z) < 1, \quad z > 0.$$

Lemma 2.2. If $0 < \alpha \le 1$ and $\beta \ge \alpha$. Then the generalized Mittag-Leffler function $E_{\alpha,\beta}(-z)$, $z \ge 0$, is completely monotonic, that is,

$$(-1)^n \frac{d^n}{dx^n} E_{\alpha,\beta}(-z) \ge 0, \text{ for } z \ge 0 \text{ and } n = 0,1,2,\ldots.$$

Proof. The proof for this result can be found in the references [17].

2.2. **Fractional calculus.** In this subsection, we compile several results from fractional calculus.

Lemma 2.3. [12] Let $\alpha \in (0,1)$ and $v = v(t) \in C[0,T]$ with $D_t^\alpha v \in C[0,T]$. Then the following statements hold:

(a) If $t_0 \in (0,T]$ and $v(t_0) = \max_{t\in[0,T]} v(t)$, then we have $D_t^\alpha v \ge 0$.

(b) If $t_0 \in (0,T]$ and $v(t_0) = \min_{t\in[0,T]} v(t)$, then we have $D_t^\alpha v \le 0$.

Lemma 2.4. The Riemann-Liouville integral I_t^α of order $\alpha > 0$ is defined as

$$I_t^\alpha v = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} v(\tau) d\tau.$$

For $0 < \alpha < 1$, $v(t)$ and $D_t^\alpha v(t)$ are both continuous on the interval $[0,T]$, then we have

$$\left(D_t^\alpha \circ I_t^\alpha v\right)(t) = v(t), \left(I_t^\alpha \circ D_t^\alpha v\right)(t) = v(t) - v(0), \ t \in [0,T].$$

2.3. **Generalized mean value theorem.** In this part, we present the generalized mean value theorem.

Theorem 2.5. [13] Let $0 < \alpha \le 1$, $a < b$ and $g \in C[a,b]$ be such that ${}_aD_t^\alpha g \in C[a,b]$. Then, there exists some $t \in (a,b)$ such that

$$g(b) - g(a) = \frac{1}{\Gamma(\alpha+1)} {}_aD_t^\alpha g(t) \cdot (b-a)^\alpha.$$

Here ${}_aD_t^\alpha$ is the Caputo derivative defined by

$${}_aD_t^\alpha g = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} \frac{d}{d\tau} g(\tau) d\tau.$$

Theorem 2.5 plays a crucial role in establishing the existence of a solution to our inverse problems.

3. FORMULATION OF THE PROBLEM AND AUXILARY LEMMAS

In a rectangular domain $\Omega = \{(x,t) : 0 < x < 1, 0 < t < T\}$, we consider the following equation

(3.1) $$ {}_C D_{0t}^{\alpha} u(x,t) + \left[x^{\beta} u_{xx}(x,t) \right]_{xx} = 0, $$

where ${}_C D_{0t}^{\alpha}$ is Caputo fractional differential operator α order [14]

$$ {}_C D_{0t}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(z)dz}{(t-z)^{\alpha-n+1}}, \left(n = \left[\mathrm{Re}(\alpha) \right] + 1, t > 0 \right) $$

α, β, T are given real numbers, such that $0 < \alpha \le 1,\ 0 \le \beta < 1.$

3.1. **Inverse problem.** Show the existence and uniqueness of functions $\{u(x,t), \varphi(x)\}$ with the following properties:

1. $u(x,t) \in C(\bar{\Omega}), \left[x^{\beta} u_{xx}(x,t) \right]_{xx}$, ${}_C D_{0t}^{\alpha} u(x,t) \in C(\Omega)$;

2. They satisfy equation (1) in the domain Ω;

3. $u(x,t)$ satisfies the following initial condition:

(3.2) $$ u(x,0) = u(x,\xi) + \varphi(x),\ x \in [0,1],\ \xi \in (0,T) $$

and the following boundary conditions:

(3.3) $$ u(0,t) = 0,\ u(1,t) = 0,\ u_x(1,t) = 0,\ \lim_{x \to 0} x^{\beta} u_{xx}(x,t) = 0,\ 0 < t \le T, $$

and the following over-determination condition

(3.4) $$ u(x,T) = \psi(x),\ x \in [0,1] $$

where $\psi(x)$ is a given function continuous on $[0,1]$, such that $\psi(0) = 0$, $\psi(1) = 0$, $\psi'(1) = 0$, $\lim_{x \to 0} x^{\beta} \psi''(x) = 0$.

Firstly, we consider homogeneous equation

$$ {}_C D_{0t}^{\alpha} u(x,t) = -\left[x^{\beta} u_{xx}(x,t) \right]_{xx}, (x,t) \in \Omega. $$

We seek a solution of this homogeneous equation satisfying the condition (3.3) in the form $u(x,t) = v(x)T(t)$. Then, with respect to the function $T(t)$, we have the following equation

$$ {}_C D_{0t}^{\alpha} T(t) - \lambda T(t) = 0, $$

and with respect to $v(x)$, we obtain the following spectral problem:

(3.5) $$ Lv \equiv \left[x^{\beta} v''(x) \right]'' = \lambda v(x),\ x \in (0,1), $$

(3.6) $$ v(0) = 0\ ,\ v(1) = 0,\ v'(1) = 0,\ \lim_{x \to 0} x^{\beta} v''(x) = 0, $$

i.e. a problem of finding those values of the parameter λ for which there exists a nontrivial solution of the equation (3.5) satisfying the condition (3.6).

3.2. **Analyzing spectral problem.** Let's investigate problem {(3.5),(3.6)}. Assume that there exist eigenvalues of the spectral problem {(3.5),(3.6)}. Under this assumption, first, we will define the sign of the eigenvalue λ. To this end, by multiplying both sides of (3.5) to the function $v(x)$ and integrating on $[0,1]$, we obtain

$$\lambda\int_0^1 v^2(x)dx=\int_0^1\left[x^\beta v''(x)\right]''v(x)dx.$$

Using the rule of integration by parts two times on the right-hand side of the last equality, we get

$$\lambda\int_0^1 v^2(x)dx=\left[\left[x^\beta v''(x)\right]'v(x)-x^\beta v''(x)v'(x)\right]\Big|_0^1+\int_0^1 x^\beta\left[v''(x)\right]^2dx.$$

Hence, considering the condition (3.6), we have

$$\lambda\int_0^1 v^2(x)dx=\int_0^1 x^\beta\left[v''(x)\right]^2dx,$$

from which it follows that $\lambda\geq 0$.

Let $\lambda=0$. Then, form (3.5), we have $\left[x^\beta v''(x)\right]''=0$. It is easy to see that the general solution of this equation has the following form

$$v(x)=C_1\frac{x^{3-\beta}}{(2-\beta)(3-\beta)}+C_2\frac{x^{2-\beta}}{(1-\beta)(2-\beta)}+C_3x+C_4,$$

where C_j, $j=\overline{1,4}$ are arbitrary constants.

Satisfying this function to the conditions (3.6), we get

$$C_4=0,\ C_2=0,\ \frac{C_1}{(2-\beta)(3-\beta)}+C_3=0,\ \frac{C_1}{2-\beta}+C_3=0,$$

from which it follows that $C_1=C_2=C_3=C_4=0$. Hence, $v(x)\equiv 0$. From the above proved that problem {(3.5),(3.6)} can have nontrivial solution only for $\lambda>0$.

Now, we will show the existence of the eigenvalues and eigenfunctions of the problem {(3.5),(3.6)}. In order to do this, we construct Green's function of the problem {(3.5),(3.6)}. It is unique and represents as follows:

$$G(x,s)=\begin{cases}\dfrac{x^{3-\beta}}{(2-\beta)}\left(-\dfrac{s^{3-\beta}}{(2-\beta)(3-\beta)}+\dfrac{s}{2-\beta}-\dfrac{1}{3-\beta}\right)+\dfrac{x}{(2-\beta)}\cdot \\ \cdot\left(\dfrac{s^{3-\beta}}{2-\beta}-\dfrac{s^{2-\beta}}{1-\beta}+\dfrac{s}{(1-\beta)(2-\beta)}\right), & x<s, \\ \dfrac{s^{3-\beta}}{(2-\beta)}\left(-\dfrac{x^{3-\beta}}{(2-\beta)(3-\beta)}+\dfrac{x}{2-\beta}-\dfrac{1}{3-\beta}\right)+\dfrac{s}{(2-\beta)}\cdot \\ \cdot\left(\dfrac{x^{3-\beta}}{2-\beta}-\dfrac{x^{2-\beta}}{1-\beta}+\dfrac{x}{(1-\beta)(2-\beta)}\right), & x>s.\end{cases}$$

Then the problem {(3.5),(3.6)} will be equivalent to the integral equation [14]

(3.7) $$v(x)=\lambda\int_0^1 G(x,s)v(s)ds.$$

Since $G(x,s)$ is symmetric and continuous, then from the theory of integral equations with symmetric kernels [15] it follows that equation (3.7) [hence problem {(3.5),(3.6)}] has a countable set of eigenvalues

$$0<\lambda_1<\lambda_2<\lambda_3<...$$

and corresponding orthonormal eigenfunctions

$$v_1(x),v_2(x),v_3(x),...,$$

and, also any arbitrary function $g(x)\in L_2[0,1]$ expands into a series in these eigenfunctions, which converges on average.

3.3. **Auxiliary lemmas.** Let $\{v_k(x)\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenfunctions and eigenvalues of the spectral problem {(3.5),(3.6)}, respectively. In our previous work [16], we proved the following lemmas:

Lemma 3.1. On the interval $[0,1]$ the following series converge uniformly:

$$\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left[v_k'(x)\right]^2}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left[x^{\beta}v_k''(x)\right]^2}{\lambda_k^2},\ \sum_{k=1}^{+\infty}\frac{\left\{[x^{\beta}v_k''(x)]'\right\}^2}{\lambda_k^2}.$$

Lemma 3.2. Let the following conditions be fulfilled*:*

$$f(x),\ f'(x)\in C[0,1],\ x^{\beta/2}f''(x)\in L_2[0,1],\ f(0)=0,\ f(1)=0,\ f'(1)=0.$$

Then the following Bessel type inequality holds:

(3.8) $$\sum_{k=1}^{\infty}\lambda_k f_k^2\le\int_0^1 x^{\beta}\left[f''(x)\right]^2dx.$$

Particularly, it is possible to assert convergence of the series in (3.8).

Here in after f_k denotes Fourier coefficient of the function $f(x)$ by the system of eigenfunctions $\{v_k(x)\}_{k=1}^{+\infty}$.

Lemma 3.3. Suppose that the following conditions hold:

$$f(x),\ f'(x),\ x^{\beta}f''(x),\ \left[x^{\beta}f''(x)\right]' \in C[0,1],\ Lf(x) \in L_2[0,1],$$

$$f(0)=0,\ f(1)=0,\ f'(1)=0,\ \lim_{x\to 0} x^{\beta}f''(x)=0.$$

Then the following Bessel type inequality holds:

$$\sum_{k=1}^{\infty}\lambda_k^2 f_k^2 \le \int_0^1 \left[\left(x^{\beta}f''(x)\right)''\right]^2 dx. \tag{3.9}$$

Particularly, it is possible to assert convergence of the series in (3.9).

Lemma 3.4. If the conditions

$$f'(x),\left[Lf(x)\right]' \in C[0,1],\quad x^{\beta/2}Lf(x) \in L_2(0,2),$$

$$f(0)=0,\ f(1)=0,\ f'(1)=0,\ \lim_{x\to 0} x^{\beta}f''(x)=0,$$

$$Lf(x)\big|_{x=0}=0,\ Lf(x)\big|_{x=1}=0,\ \left[Lf(x)\right]'\Big|_{x=1}=0,$$

then the following inequality holds:

$$\sum_{k=1}^{\infty}\lambda_k^3 f_k^2 \le \int_0^1 x^{\beta}\left\{\left[Lf(x)\right]''\right\}^2 dx. \tag{3.10}$$

Lemma 3.5. Suppose that the following conditions hold:

$$f''(x),\left[x^{\beta}\left[Lf(x)\right]''\right]' \in C[0,1],\ L^2 f(x) \in L_2[0,1],$$

$$f(0)=0,\ f(1)=0,\ f'(1)=0,\ \lim_{x\to 0} x^{\beta}f''(x)=0,$$

$$Lf(x)\big|_{x=0}=0,\ Lf(x)\big|_{x=1}=0,\ \left[Lf(x)\right]'\Big|_{x=1}=0,\ x^{\beta}\left[Lf(x)\right]''\Big|_{x=0}=0.$$

Then the following Bessel type inequality holds:

$$\sum_{k=1}^{\infty}\lambda_k^5 f_k^2 \le \int_0^1 x^{\beta}\left(M^2 f(x)\right)^2 dx. \tag{3.11}$$

Proof. Using the equation (3.5), we can write

$$\lambda_k^{5/2} f_k = \lambda_k^{5/2} \int_0^1 f(x) v_k(x)\,dx = \lambda_k^{3/2} \int_0^1 f(x) L v_k(x)\,dx = \lambda_k^{1/2} \int_0^1 f(x) \left[x^\beta v_k'' \right]'' dx.$$

Hence, applying the rule of integration by part eight times and considering the properties of the functions $f(x)$ and $v_k(x)$, we get

$$\lambda_k^{5/2} f_k = \lambda_k^{1/2} \int_0^1 \left[x^\beta \left[x^\beta f''(x) \right]^{(4)} \right]'' v_k(x)\,dx = \lambda_k^{1/2} \int_0^1 L^2 f(x) v_k(x)\,dx.$$

Replacing the function $v_k(x)$ in the last integral with $\left[x^\beta v_k'' \right]'' / \lambda_k$ and then applying integration by parts to the resulting integral, we have

$$\lambda_k^{5/2} f_k = \int_0^1 x^{\beta/2} L^2 f(x) \left[\lambda_k^{-1/2} x^{\beta/2} v_k''(x) \right] dx.$$

Thus, the number $\lambda_k^{5/2} f_k$ is the Fourier coefficient of the function $x^{\beta/2} L^2 f(x)$ with respect to the orthonormal system of functions $\left\{ \lambda_k^{-1/2} x^{\beta/2} v_k''(x) \right\}_{k=1}^{+\infty}$. Then, according to Bessel's inequality, we come to the inequality (3.11).

Lemma 3.5 has been proved.

4. THE MAIN RESULT

We will expand the functions $u(x,t)$, $\varphi(x)$, $\psi(x)$ into series with respect to the system of functions $\left\{ v_k(x) \right\}_{k=1}^{\infty}$ by formulas, respectively,

(4.1) $$u(x,t) = \sum_{k=1}^{\infty} T_k(t) v_k(x)$$

and

(4.2) $$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k v_k(x),\ \psi(x) = \sum_{k=1}^{\infty} \psi_k v_k(x).$$

Substituting (4.1), (4.2) and into the equation (3.1), initial condition (3.2) and the over-determination condition (3.4), for defining $T_k(t)$ and φ_k, we will have the following problem

(4.3) $$\begin{cases} {}_C D_{0t}^{\alpha} T_k(t) + \lambda_k T_k(t) = 0 \\ T_k(0) = T_k(\xi) + \varphi_k \\ T_k(T) = \psi_k. \end{cases}$$

It is easy to verify that from the problem (4.3), $T_k(t)$ and φ_k can be found in the following forms

(4.4) $$T_k(t)=\frac{\psi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)}{E_{\alpha,1}\left(-\lambda_k T^{\alpha}\right)},$$

(4.5) $$\varphi_k=\frac{\psi_k\left(1-E_{\alpha,1}\left(-\lambda_k \xi^{\alpha}\right)\right)}{E_{\alpha,1}\left(-\lambda_k T^{\alpha}\right)}.$$

Then, the formal solution to the inverse problem is given by

(4.6) $$u(x,t)=\sum_{k=1}^{+\infty}\left[\frac{\psi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)}{E_{\alpha,1}\left(-\lambda_k T^{\alpha}\right)}\right]v_k(x),$$

(4.7) $$\varphi(x)=\sum_{k=1}^{+\infty}\left[\frac{\psi_k\left(1-E_{\alpha,1}\left(-\lambda_k \xi^{\alpha}\right)\right)}{E_{\alpha,1}\left(-\lambda_k T^{\alpha}\right)}\right]v_k(x).$$

Theorem 4.1. Assume that the function $\psi(x)$ satisfies the conditions of Lemma 3.5. Then the pair of functions defined by (4.6) and (4.7) constitutes the unique solution to the inverse problem.

Proof. We consider the series (4.6), and Lemma 2.1 we have

$$\left|u(x,t)\right|\le\sum_{k=1}^{\infty}\left|1+\Gamma(1-\alpha)\lambda_k T^{\alpha}\right|\left|\psi_k\right|\left|v_k(x)\right|\le\sum_{k=1}^{\infty}\left|\psi_k\right|\left|v_k(x)\right|+C_1\sum_{k=1}^{\infty}\left|\lambda_k\right|\left|\psi_k\right|\left|v_k(x)\right|\le$$

(4.8) $$\le\sum_{k=1}^{+\infty}\left|\lambda_k^{1/2}\psi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right|+C_1\sum_{k=1}^{+\infty}\left|\lambda_k^{3/2}\psi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right|,$$

where $C_1=\Gamma(1-\alpha)T^{\alpha}$.

Applying the Cauchy-Schwarz inequality, we get

$$\sum_{k=1}^{+\infty}\left|\lambda_k^{1/2}\psi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right|\le\left[\sum_{k=1}^{+\infty}\lambda_k\psi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|\lambda_k^{3/2}\psi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right|\le\left[\sum_{k=1}^{+\infty}\lambda_k^3\psi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

The series on the right-hand side of these inequalities converge uniformly in $x\in[0,1]$ due to Lemma 3.2 and Lemma 3.4. Therefore, the series on the left-hand side also converges uniformly in $x\in[0,1]$. Hence, from (4.8), it follows that the series (4.6) converges absolutely and uniformly in $\overline{\Omega}$.

Now, we consider the series responding to the function $\left[x^{\beta}u_{xx}(x,t)\right]_{xx}$. From (4.8), considering Lemma 2.1, we have

$$\left[x^{\beta}u_{xx}(x,t)\right]_{xx} \le \sum_{k=1}^{\infty}\left|1+\Gamma(1-\alpha)\lambda_k T^{\alpha}\right|\lambda_k|\psi_k|\|v_k(x)| \le \sum_{k=1}^{\infty}\lambda_k|\psi_k|\|v_k(x)| +$$

$$+C_1\sum_{k=1}^{\infty}\lambda_k^2|\psi_k|\|v_k(x)| \le \sum_{k=1}^{+\infty}\lambda_k^{3/2}|\psi_k|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| + C_1\sum_{k=1}^{+\infty}\lambda_k^{5/2}|\psi_k|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right|. \tag{4.9}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=1}^{+\infty}\left|\lambda_k^{3/2}\psi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \le \left[\sum_{k=1}^{+\infty}\lambda_k^3\psi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|\lambda_k^{5/2}\psi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \le \left[\sum_{k=1}^{+\infty}\lambda_k^5\psi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

The series on the right-hand side of these inequalities converge uniformly in $x\in[0,1]$ due to Lemma 3.4 and Lemma 3.5. Therefore, the series on the left-hand side also converges uniformly in $x\in[0,1]$. Hence, from (4.9), it follows that the series corresponding to the function $\left[x^{\beta}u_{xx}(x,t)\right]_{xx}$ converges absolutely and uniformly in $\bar{\Omega}$.

Now, considering the series corresponding to the function $\varphi(x)$ from (4.7), we get

$$|\varphi(x)| \le \sum_{k=1}^{\infty}\left|1+\Gamma(1-\alpha)\lambda_k T^{\alpha}\right|\|\psi_k\|v_k(x)| \le \sum_{k=1}^{\infty}|\psi_k|\|v_k(x)| + C_1\sum_{k=1}^{\infty}\lambda_k|\psi_k|\|v_k(x)| \le$$

$$\le \sum_{k=1}^{+\infty}\lambda_k^{1/2}|\psi_k|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| + C_1\sum_{k=1}^{+\infty}\lambda_k^{3/2}|\psi_k|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right|. \tag{4.10}$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum_{k=1}^{+\infty}\left|\lambda_k^{1/2}\psi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \le \left[\sum_{k=1}^{+\infty}\lambda_k\psi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|\lambda_k^{3/2}\psi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \le \left[\sum_{k=1}^{+\infty}\lambda_k^3\psi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

The series on the right-hand side of these inequalities converge uniformly in $x\in[0,1]$ due to Lemma 3.2 and Lemma 3.4. Therefore, the series on the left-hand side also converges uniformly in $x\in[0,1]$. Hence, from (4.10), it follows that the series (4.7) converges absolutely and uniformly in $\bar{\Omega}$.

Theorem 4.1 has been proved.

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