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Study of Transient Modes of Incompressible Fluid Flow in Relief Sections of Pipelines with Changes in Velocity at Their Boundaries

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Abstract: The paper considers modeling of unsteady flow of incompressible fluid in an elementary section of a pipeline with changing relief, in which the boundary conditions for velocity change over time. The study is based on linearized quasi-one-dimensional equations modified based on the approach of N.E. Zhukovsky. To analyze the flow velocity, the method of separation of variables is used, which allows solving a simplified telegraph equation. The pressure is determined by integrating the equations of motion taking into account the found velocity distribution.

Numerical modeling of transient processes was performed for various variants of initial and boundary conditions, including cases with constant functions. Scenarios of average pressure change along the section were analyzed: stable, increasing and decreasing. Particular attention was paid to the influence of friction on the formation and evolution of shock waves in comparison with the model where the friction force is not taken into account.

Keywords: quasi-one-dimensional model, pipeline, resistance force, slightly compressible fluid, Fourier method, gravity, computational experiment.

INTRODUCTION

In engineering practice, the calculation of pipeline systems is most often carried out on the basis of stationary flow regimes, ignoring the influence of local instabilities [1]. However, when analyzing non-stationary processes, weak points in the structure of the pipeline network become noticeable. One of the characteristic manifestations of non-stationarity are transient processes that occur when regimes change. Such processes are accompanied by phenomena that require taking into account both the movement of the main flow and the propagation of small disturbances in the 'pipe - transported liquid' system, which requires the analysis of two different speed scales.

The processes of generation and propagation of disturbances in pipeline systems are considered on the basis of quasi-one-dimensional models developed by N.E. Zhukovsky [2,3]. These equations made it possible for the first time to combine the description of the movement of the transported medium taking into account the hydrodynamic velocity and the speed of transmission of small disturbances in the 'pipe-liquid' system. Based on the

proposed approach, both analytical and experimental studies were conducted aimed at studying the behavior of compression and rarefaction waves in pipelines.

Progress in the field of pipeline transport is largely due to the development of theoretical foundations for modeling such systems. In this regard, both linear and nonlinear mathematical descriptions of the processes of movement of low- and super-compressible [4], as well as hydrodynamic mixtures, modeled within the framework of Newtonian and non-Newtonian media [5] were proposed. Analytical [6,7], numerical [8,9] and approximate methods were used to analyze both individual elements and complex pipeline networks, taking into account various physical and energy influences or excluding them depending on the modeling tasks.

Power plants of pipeline systems (pumps, compressors) operate within strictly defined limits of input pressure and flow rate of the medium. A decrease in these parameters can lead to the phenomenon of surge , accompanied by intense vibrations in the transported medium. These vibrations are then transmitted to the structure of the pipeline and pumping equipment. The coincidence of the natural frequencies of the pipeline-medium system can cause resonance, which in turn can lead to damage to the pump and pipeline.

During the process of starting and stopping pumping stations, pressure waves of disturbances are generated, which can lead to significant damage in the pipeline system [10].

Taking into account the characteristics of shock waves and the requirements of engineering practice, an elementary section of a pipeline with relief will be considered in what follows, where the change in velocity over time is specified at both ends. The initial conditions include a known distribution of velocity and its derivatives over time. To solve the problem, the Fourier method will be used within the framework of N.E. Zhukovsky's quasi-one-dimensional model.

MATERIALS AND METHODS

A general approach to solving the problem is proposed, taking into account force factors such as friction, gravity, and local inertial forces at variable input and output velocities. Numerical results were obtained for constant values of the functions defining the initial and boundary conditions. Unlike traditional solutions, in this case the relief of the pipeline route is taken into account.

The dynamic state of the liquid in an elementary section of the pipeline is $\rho \approx const$ described by the equations [2,11]:

$$\begin{cases} -\dfrac{\partial P}{\partial x} = \rho\left(\dfrac{\partial w}{\partial t} + 2aw\right), \\ -\dfrac{\partial P}{\partial t} = \rho c^2 \dfrac{\partial w}{\partial x}. \end{cases}$$

Here $P(x,t) = p(x,t) + \rho g y(x)$ – total hydrostatic pressure, $y(x)$ – leveling height of the pipeline axis; $a = \dfrac{\lambda w_*}{4D}$; $p(x,t)$, $w(x,t)$ – average x values of pressure and flow velocity over the cross-section at the moment of time t; w_* – characteristic process velocity – averaging parameter; λ – coefficient of friction resistance; $c = \left(\dfrac{\rho_0}{k} + \dfrac{D\rho_0}{E\delta}\right)^{-1/2}$ – speed

of propagation of small pressure disturbances in the pipe liquid system; k, E – elastic moduli of the transported liquid and pipe; δ – pipe thickness, ρ_0, D – the density of the liquid and the diameter of the pipeline in an undisturbed state [11].

Elimination of full pressure $P(x,t)$ at constant cross-sectional area of the pipe $f = \pi D^2 / 4$ allows us to construct a separate equation for the velocity:

$$\frac{\partial^2 w}{\partial t^2} + 2a\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

This is the incomplete telegraph equation [17,16].

The following variant of initial conditions is considered:

$$w(x,0) = w^{(0)}(x),\ \frac{\partial w(x,0)}{\partial t} = w^{(t)}(x).$$

The boundary conditions are given in the form

$$w(0,t) = \bar{w}(t),\ w(l,t) = \tilde{w}(t).$$

According to these data and the first equation of the original system, the initial condition for the total pressure is

$$P(x,0) = p_{00} - \rho\int_0^x \left[w^{(t)}(\eta) + 2aw^{(0)}(\eta) \right] d\eta,$$

where p_{00} is the initial pressure in the inlet section.

Let us bring the boundary conditions to a homogeneous form. For this purpose, a new desired function is introduced $u(x,t)$ according to the dependence

$$w(x,t) = C_0(t) + xC_1(t) + u(x,t).$$

The values of the functions are determined

$$C_0(t) = \bar{w}(t),\ C_1(t) = \frac{\tilde{w}(t) - \bar{w}(t)}{l},$$

which allow us to represent the boundary conditions in a homogeneous form:

$$u(0,t) = 0,\ u(l,t) = 0.$$

In this case, the equation itself becomes non-homogeneous.

$$\frac{\partial^2 u}{\partial t^2} + 2a\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} - H(t) - xR(t).$$

It is solved under new initial conditions.

$$u(x,0) = w^{(0)}(x) - C_0(0) - C_1(0)x,$$

$$\frac{\partial u(x,0)}{\partial t} = w^{(t)}(x) - C_0'(0) - C_1'(0)x$$

and under already known homogeneous boundary conditions.

From here on we use the notations

$$H(t) = C_0''(t) + 2aC_0'(t), \quad R(t) = C_1''(t) + 2aC_1'(t).$$

Let us consider the solution of the equation for the auxiliary function. The linear nature of the equations and boundary conditions allows us to simplify the analysis process. In this formulation, the problem is solved by representing the general solution as the sum of two partial solutions obtained by the superposition method [11,12].

The first term $U(x,t)$ is the general solution of the homogeneous equation

$$\frac{\partial^2 U}{\partial t^2} + 2a\frac{\partial U}{\partial t} = c^2\frac{\partial^2 U}{\partial x^2}$$

at given boundary conditions $U(0,t) = 0,\ U(l,t) = 0$ and initial conditions

$$U(x,0) = w^{(0)}(x) - C_0(0) - C_1(0)x,$$

$$\frac{\partial U(x,0)}{\partial t} = w^{(t)}(x) - C_0'(0) - C_1'(0)x.$$

The second term $V(x,t)$ represents a particular solution of the inhomogeneous equation in accordance with the right-hand side

$$\frac{\partial^2 V}{\partial t^2} + 2a\frac{\partial V}{\partial t} = c^2\frac{\partial^2 V}{\partial x^2} - H(t) - xR(t),$$

which is solved with zero initial values

$$V(x,0) = 0,\ \frac{\partial V(x,0)}{\partial t} = 0.$$

and homogeneous boundary conditions

$$V(0,t) = 0,\ V(l,t) = 0.$$

First, the homogeneous equation is solved.

Let's assume that

$$U(x,t) = X(x)Y(t),$$

and we arrive at the equations:

$$\frac{Y''(t) + 2aY'(t)}{c^2 Y(t)} = \frac{X''(x)}{X(x)} = -\lambda^2.$$

Liouville problem for the coordinate x gives [11,12,13]

$$X_n(x) = \sin\lambda_n x,\ \lambda_n = \frac{n\pi}{l},\ \|X_n(x)\|^2 = \frac{l}{2}.$$

In this case, to determine the multiplier in the form of an eigenfunction of time, we have the equation

$$Y_n''(t) + 2aY_n'(t) + c^2\lambda_n^2 Y_n(t) = 0.$$

Substituting the expression $Y_n(t) = e^{s_n t}$ into this equation will yield the characteristic equation

$$s_n^2 + 2as_n + c^2\lambda_n^2 = 0.$$

The discriminant of this equation is

$$D_n = a^2 - c^2\lambda_n^2,$$

and the solutions of the quadratic equation will be:

$$(s_n)_{1,2} = -a \pm \sqrt{D_n}.$$

For $D_n > 0$, when the two roots of the characteristic equation are real and different, the solution of the differential equation for $Y_n(t)$ will be [14]:

$$Y_n(t) = A_n' e^{-at+\sqrt{D_n}t} + B_n' e^{-at-\sqrt{D_n}t} = e^{-at}\left(A_n' e^{\sqrt{D_n}t} + B_n' e^{-\sqrt{D_n}t}\right) =$$

$$= e^{-at}\left(\frac{A_n + B_n}{2} e^{\sqrt{D_n}t} + \frac{A_n + B_n}{2} e^{-\sqrt{D_n}t}\right) = e^{-at}\left(A_n ch\sqrt{D_n}t + B_n sh\sqrt{D_n}t\right).$$

When $D_n = 0$ the solutions of the characteristic equation are the same (double root) and the solution of the differential equation $Y_n(t)$ will be [14]:

$$Y_n(t) = e^{-at}\left(A_n + B_n t\right).$$

When $D_n < 0$ solving the characteristic equation, there will be conjugate complex equations: $(s_n)_{1,2} = -a \pm i\sqrt{|D_n|}$. Therefore, the solution of the differential equation by $Y_n(t)$ will be [15].

$$Y_n(t) = A_n' e^{-at+i\sqrt{|D_n|}t} + B_n' e^{-at-i\sqrt{|D_n|}t} = e^{-at}\left(A_n' e^{i\sqrt{|D_n|}t} + B_n' e^{-i\sqrt{|D_n|}t}\right) =$$

$$= e^{-at}\left(\frac{A_n - iB_n}{2} e^{i\sqrt{|D_n|}t} + \frac{A_n + iB_n}{2} e^{-i\sqrt{|D_n|}t}\right) = e^{-at}\left(A_n chi\sqrt{|D_n|}t - iB_n shi\sqrt{|D_n|}t\right) =$$

$$= e^{-at}\left(A_n \cos\sqrt{|D_n|}t + B_n \sin\sqrt{|D_n|}t\right).$$

Taking these possible options into account $D_n > 0$, $D_n = 0$ and $D_n < 0$ for $Y_n(t)$ we get the solution:

$$Y_n(t) = \begin{cases} e^{-at}\left(A_n ch\sqrt{D_n}t + B_n sh\sqrt{D_n}t\right) \text{ при } D_n > 0, \\ e^{-at}\left(A_n + B_n t\right) \text{ при } D_n = 0, \\ e^{-at}\left(A_n \cos\sqrt{|D_n|}t + B_n \sin\sqrt{|D_n|}t\right) \text{ при } D_n < 0. \end{cases}$$

Accordingly, the general solution of the homogeneous equation is

$$U(x,t)=\sum_{n=1}^{\infty}Y_n(t)\sin\lambda_n x.$$

From the first initial condition (at $t=0$) it follows:

$$\sum_{n=1}^{\infty}A_n\sin\lambda_n x=w^{(0)}(x)-C_0(0)-xC_1(0).$$

We multiply the sides of this equation by $\sin\lambda_m x dx$ and integrate from zero to l. As a result, we get

$$A_n\left\|X_n(x)\right\|^2=W_n-C_0(0)S_n-C_1(0)K_n,$$

Where

$$W_n=\int_0^l w^{(0)}(x)\sin\lambda_n x dx,$$

$$S_n=\int_0^l\sin\lambda_n x dx=\frac{l}{n\pi}\left[1-(-1)^n\right],$$

$$K_n=\int_0^l x\sin\lambda_n x dx=\frac{l^2}{n\pi}(-1)^{n+1}.$$

From here we determine

$$A_n=\frac{2}{l}\left[W_n-C_0(0)S_n-C_1(0)K_n\right].$$

The second initial condition is related to the time derivative of the function $U(x,t)$ at $t=0$.

We find

$$Y_n'(t)=\begin{cases}e^{-at}\left[\left(-aB_n+\sqrt{D_n}A_n\right)sh\sqrt{D_n}t+\left(-aA_n+\sqrt{D_n}B_n\right)ch\sqrt{D_n}t\right]\ \text{при}\ D_n>0,\\ e^{-at}\left[-a\left(A_n+B_nt\right)+B_n\right]\ \text{при}\ D_n=0,\\ e^{-at}\left[\left(-aB_n-\sqrt{|D_n|}A_n\right)\sin\sqrt{D_n}t+\left(-aA_n+\sqrt{|D_n|}B_n\right)\cos\sqrt{D_n}t\right]\ \text{при}\ D_n<0.\end{cases}$$

Let's calculate

$$Y_n'(0)=\left\{\begin{array}{l}-aA_n+\sqrt{D_n}B_n\ \ при\ \ D_n>0,\\ -aA_n+B_n\ \ при\ \ D_n=0,\\ -aA_n+\sqrt{|D_n|}B_n\ \ при\ \ D_n<0\end{array}\right\}=-aA_n+\gamma_nB_n,$$

Where $\gamma_n = \begin{cases} \sqrt{D_n} & \text{при } D_n > 0, \\ 1 & \text{при } D_n = 0, \\ \sqrt{|D_n|} & \text{при } D_n < 0. \end{cases}$

Then the second initial condition takes the form:

$$\sum_{n=1}^{\infty}(-aA_n + \gamma_n B_n)\sin\lambda_n x = w^{(t)}(x) - C_0'(0) - xC_1'(0).$$

The application of orthonormal eigenfunctions with respect to x, for $W_n^{(t)} = \int_0^t w^{(t)}(\eta)\sin\lambda_n \eta d\eta$, leads to the equality

$$(-aA_n + \gamma_n B_n)\frac{l}{2} = W_n^{(t)} - C_0'(0)S_n - C_1'(0)K_n.$$

From here we find the values of the coefficients

$$B_n = \frac{a}{\gamma_n}A_n + \frac{2}{\gamma_n l}\left[W_n^{(t)} - C_0'(0)S_n - C_1'(0)K_n\right].$$

Substituting this solution into the inhomogeneous equation leads to the equation

$$\sum_{n=1}^{\infty}\left[L_n''(t) + 2aL_n'(t) + c^2\lambda_n^2 L_n(t)\right]\sin\lambda_n x = -H(t) - xR(t).$$

We use the conditions of orthonormal eigen functions $X_n(x) = \sin\lambda_n x$:

$$L_n''(t) + 2aL_n'(t) + c^2\lambda_n^2 L_n(t) = \frac{2}{l}\left[-H(t)S_n - R(t)K_n\right].$$

When $t = 0$ we have the condition $L_n(0) = 0.$ From the second initial condition $\frac{\partial V(x,0)}{\partial t} = 0$ it follows that $L_n'(0) = 0$.

In general, cases are considered where functions $L_n(t)$ are defined analytically. This class includes cases of linear, exponential, trigonometric and some other forms of specifying boundary conditions for velocity. In this case, the final solution for the auxiliary function $u(x,t)$ is defined as

$$u(x,t) = \sum_{n=1}^{\infty}\left[Y_n(t) + L_n(t)\right]\sin\lambda_n x.$$

Going back, we get a solution to the problem in terms of speed

$$w(x,t) = C_0(t) + xC_1(t) + \sum_{n=1}^{\infty}\left[Y_n(t) + L_n(t)\right]\sin\lambda_n x.$$

We find the solution to the problem regarding pressure by integrating the initial system of quasi-one-dimensional equations.

In the first step of the solution , $P(x,t)$ we turn to the second equation $\frac{\partial P}{\partial t} = -\rho c^2 \frac{\partial w}{\partial x}$ of the original system.

We find

$$\frac{\partial w(x,t)}{\partial x} = C_1(t) + \sum_{n=1}^{\infty} \lambda_n \left[Y_n(t) + L_n(t) \right] \cos \lambda_n x \, .$$

As a result of integrating the second equation of the original system, we obtain the formula for pressure:

$$P(x,t) = -\rho c^2 \left[C_1^{(0)}(t) + \sum_{n=1}^{\infty} \lambda_n \left[Y_n^{(0)}(t) + L_n^{(0)}(t) \right] \cos \lambda_n x \right] + p_t(x).$$

Here

$\int_0^t C_1(\theta) dt = C_1^{(0)}(t)$ And $\int_0^t L_n(\theta) d\theta = L_n^{(0)}(t)$,

$$Y_n^{(0)}(t) = \int_0^t Y_n(\theta) d\theta = \begin{cases} \frac{1}{c^2 \lambda_n^2} \left[\left(-A_n a - B_n \sqrt{D_n} \right) \left(e^{-at} ch \sqrt{D_n} t - 1 \right) + \right. \\ \left. + \left(-A_n \sqrt{D_n} - B_n a \right) e^{-at} sh \sqrt{D_n} t \right] \quad \text{при } D_n > 0, \\ A_n \frac{1 - e^{at}}{a} + B_n \left(-\frac{t e^{-at}}{a} + \frac{1 - e^{-at}}{a^2} \right) \quad \text{при } D_n = 0, \\ \frac{1}{c^2 \lambda_n^2} \left[\left(-A_n a - B_n \sqrt{|D_n|} \right) \left(e^{-at} \cos \sqrt{|D_n|} t - 1 \right) + \right. \\ \left. + \left(A_n \sqrt{|D_n|} - B_n a \right) e^{-at} \sin \sqrt{|D_n|} t \right] \quad \text{при } D_n < 0. \end{cases}$$

When using the solution for the case $a \to 0$ (when the friction force is negligible), the expression $Y_n^{(0)}(t)$ must take into account the values of the limits

$$e^{-at} \to 1 \, , \; \frac{1 - e^{-at}}{a} \to t \, , \; -\frac{t e^{-at}}{a} + \frac{1 - e^{-at}}{a^2} \to t^2$$

and accept

$$\lim_{a \to 0} \left[A_n \frac{1 - e^{at}}{a} + B_n \left(-\frac{t e^{-at}}{a} + \frac{1 - e^{-at}}{a^2} \right) \right] = A_n t + B_n t^2 .$$

It is necessary to determine the value $p_t(x)$ of the constant of integration over time. To determine it, we turn to the first equation $-\frac{\partial P}{\partial x} = \rho \left(\frac{\partial w}{\partial t} + 2aw \right)$ of the original system.

Let's calculate

$$\frac{\partial P(x,t)}{\partial x} = \rho c^2 \sum_{n=1}^{\infty} \lambda_n^2 \left[Y_n^{(0)}(t) + L_n^{(0)}(t) \right] \sin \lambda_n x + p_t'(x)$$

and

$$\frac{\partial w(x,t)}{\partial t} = C_0'(t) + xC_1'(t) + \sum_{n=1}^{\infty} \left[Y_n'(t) + L_n'(t) \right] \sin \lambda_n x \, .$$

We insert these expressions and the expression for velocity $w(x,t)$ into the first equation of the original system:

$$-\rho c^2 \sum_{n=1}^{\infty} \lambda_n^2 \left[Y_n^{(0)}(t) + L_n^{(0)}(t) \right] \sin \lambda_n x - p_t'(x) =$$

$$= \rho \left[C_0'(t) + xC_1'(t) + \sum_{n=1}^{\infty} \left[Y_n'(t) + L_n'(t) \right] \sin \lambda_n x \right] +$$

$$+2a\rho \left[C_0(t) + xC_1(t) + \sum_{n=1}^{\infty} \left[Y_n(t) + L_n(t) \right] \sin \lambda_n x \right].$$

From here we find

$$p_t'(x) = -\rho \left[C_0'(t) + 2aC_0(t) \right] - \rho \left[C_1'(t) + 2aC_1(t) \right] x -$$

$$-\rho \sum_{n=1}^{\infty} \left[Y_n'(t) + 2aY_n(t) + c^2 \lambda_n^2 Y_n^{(0)}(t) \right] \sin \lambda_n x -$$

$$-\rho \sum_{n=1}^{\infty} \left[L_n'(t) + 2aL_n(t) + c^2 \lambda_n^2 L_n^{(0)}(t) \right] \sin \lambda_n x.$$

Let's transform the first sum

$$\sum_{n=1}^{\infty} \left[Y_n'(t) + 2aY_n(t) + c^2 \lambda_n^2 Y_n^{(0)}(t) \right] \sin \lambda_n x =$$

$$= \sum_{n=1}^{\infty} \int_0^t \left[Y_n''(\theta) + 2aY_n'(\theta) + c^2 \lambda_n^2 Y_n(\theta) \right] d\theta \sin \lambda_n x = \sum_{n=1}^{\infty} c_n \sin \lambda_n x d\theta = E(x).$$

Here we took into account that $\int_0^t \left[Y_n''(\theta) + 2aY_n'(\theta) + c^2 \lambda_n^2 Y_n(\theta) \right] d\theta = c_n = const$ represents the coefficients of the expansion in eigenfunctions of a certain function $E(x)$.

We will do the same with the second sum (see (3)):

$$\sum_{n=1}^{\infty} \left[L_n'(t) + 2aL_n(t) + c^2 \lambda_n^2 L_n^{(0)}(t) \right] \sin \lambda_n x =$$

$$= \sum_{n=1}^{\infty} \int_0^t \left[L_n''(\theta) + 2aL_n'(\theta) + c^2 \lambda_n^2 L_n(\theta) \right] d\theta \, \sin \lambda_n x =$$

$$= \int_0^t \frac{2}{l}\left[-H(\theta)S_n - R(\theta)K_n\right]d\theta = \frac{2}{l}\left[-H^{(0)}(t)S_n - R^{(0)}K_n\right].$$

Here $H^{(0)}(t) = \int_0^t H(\theta)d\theta, \quad R^{(0)}(t) = \int_0^t R(\theta)d\theta.$

When substituting the obtained values of the sums, we have

$$p_t'(x) = -\rho\left[C_0'(t) + 2aC_0(t)\right] - \rho\left[C_1'(t) + 2aC_1(t)\right]x -$$
$$-\rho E(x) + \rho\frac{2}{l}\left[H^{(0)}(t)S_n + R^{(0)}K_n\right] =$$
$$= -\rho\left[C_0'(t) + 2aC_0(t) - \frac{2}{l}\left(H^{(0)}(t)S_n + R^{(0)}(t)K_n\right)\right] -$$
$$-\rho\left[C_1'(t) + 2aC_1(t)\right]x + \rho E(x).$$

Now we integrate over x:

$$p_t(x) = -\rho\left\{C_0'(t) + 2aC_0(t) - \frac{2}{l}\left[H^{(0)}(t)S_n + R^{(0)}(t)K_n\right]\right\}x -$$
$$-\rho\left[C_1'(t) + 2aC_1(t)\right]\frac{x^2}{2} + \rho\int_0^x E(\eta)d\eta.$$

Let's insert the resulting expression into the formula for pressure:

$$P(x,t) = \rho c^2 C_n^{(0)}(t) - \rho\left\{C_0'(t) + 2aC_0(t) - \frac{2}{l}\left[H^{(0)}(t)S_n + R^{(0)}(t)K_n\right]\right\}x -$$
$$-\rho\left[C_1'(t) + 2aC_1(t)\right]\frac{x^2}{2} + \rho c^2\sum_{n=1}^{\infty}\lambda_n\left[Y_n^{(0)}(t) + L_n^{(0)}(t)\right]\cos\left(\lambda_n x + \varphi_n\right) +$$
$$+\rho\int_0^x E(\eta)d\eta.$$

The value of the last integral is determined according to the initial pressure distribution along the length of the section, which is presented above.

When $t = 0$ the terms highlighted by a superscript (0) and capital letters are converted to zero. In this connection, to determine the value of the integral, we obtain the equality:

$$p_{00} - \rho\int_0^x\left[w^{(t)}(\eta) + 2aw^{(0)}(\eta)\right]d\eta =$$
$$= -\rho\left[C_0'(t) + 2aC_0(t)\right]x - \rho\left[C_1'(t) + 2aC_1(t)\right]\frac{x^2}{2} + \rho\int_0^x E(\eta)d\eta.$$

From here we find the value of the integral and insert it into the formula for pressure:

$$P(x,t) = p_{00} - \rho\int_0^x \left[w^{(t)}(\eta) + 2aw^{(0)}(\eta) \right] d\eta -$$

$$-\rho c^2 \left\{ C_1^{(0)}(t) + \sum_{n=1}^{\infty} \lambda_n \left[Y_n^{(0)}(t) + L_n^{(0)}(t) \right] \cos \lambda_n x \right\}.$$

Here $C_1^{(0)}(t) = \frac{1}{l}\int_0^t \left[\bar{w}(\theta) - \tilde{w}(\theta) \right] d\theta.$

Thus, the functions for the boundary velocities $\bar{w}(t)$, $\tilde{w}(t)$ are twice differentiable and integrable functions, and the functions for $w^{(0)}(x)$ and $w^{(t)}(x)$ are integrable.

Within the framework of the presentation, a formal approach to solving the problem was presented. In this case, the derivatives and integrals included in the equations were considered as given quantities. However, the main attention is paid to the analysis of transient processes, and the calculations are limited to this class of problems. For each specific case, the corresponding numerical values of the derivatives and integrals will be given.

DISCUSSION OF RESULTS

A program has been developed to calculate the velocity and pressure in transient processes on a horizontal section of a pipeline, provided that the initial and boundary conditions are specified by constant values. The following values were used in the calculations: section length, section $l = 1200$ *м*, diameter of 0.2 m, coefficient of resistance, $\lambda = 0.018$, propagation speed of small disturbances $c = 1200$ *м*/*с*, of the input pressure, $p_{00} = 7.0$ *МПа*, scale of velocity $w_* = 5$ *м*/*с* and step along the length. $l/50$. A series of calculations were carried out with and without taking into account the friction force and in time steps $l/(100c)$ and $l/(10c)$.

The calculation was carried out using $w_{00} = 0$ *м*/*с*, $w^{(0)} = w^{(1)} = 5$ *м*/*с*. That is, in a stationary section under high pressure , liquid began to flow 7.0 *МПА* at the inlet with a speed 5 *м*/*с*, and liquid began to be taken from the outlet with a speed 5 *м*/*с*. The mass of liquid in the section remains constant.

Fig. 1 shows the velocity curves at different time steps of the first conditional period.

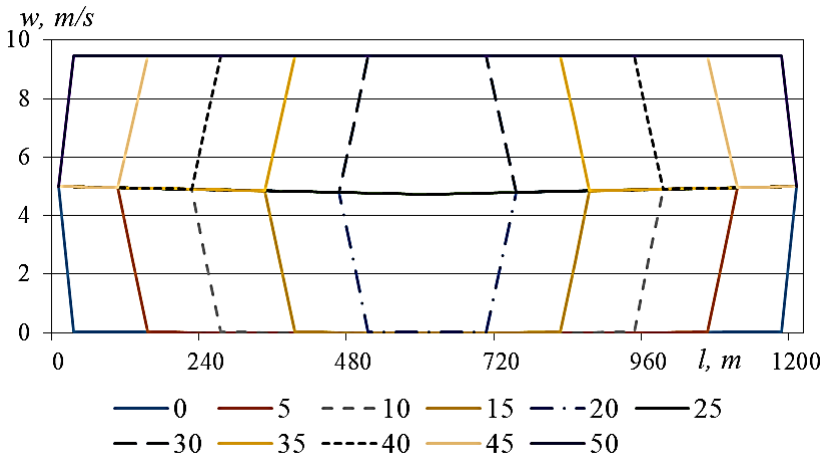

Fig. 1. Graphs of speed along the length of the section, obtained in the first ten time steps at $\tau = l/(5c)$ and $h = l/50. l = 1200\ м,\quad D = 0.2\ м,\quad \lambda = 0.018,\quad w_* = 5\ м/с,$ $c = 1200\ м/с,\ w_{00} = 0\ м/с,\ w^{(0)} = w^{(1)} = 5\ м/с,\ p_{00} = 7.00\ МПа$

In this variant, the conditional period of the process relative to the indicators is equal to l/c. The lower curves (at $t/\tau = 0,\ 5,\ 10,\ 15,\ 20$) refer to the first half of the conditional period. The process of increasing the speed begins simultaneously from both ends of the section: from zero to $5\ м/с$. The waves gradually move towards each other. At $t = 25\tau$ they mutually meet, forming an almost straight line with a constant speed $5\ м/с$. In this case, approaching the middle of the section occurs with the loss of a certain speed. Adding up these impulses, they bring the speed to a new value, which is already less than $2 \cdot 5\ м/с = 10\ м/с$. After this, the waves of compaction and rarefaction continue their movements until they reach the ends of the section. But again, the values of the speed jump decrease by a certain value.

Having reached the ends of the section, the waves turn over and strive back to a mutual meeting. Unlike the first four lower graphs, here in the lower links the speed is no longer zero, but approximately unity. When the waves return, the process is repeated, but with a smaller maximum value. And so on. In odd conditional periods, the upper envelope line of the pressure curves is transformed, and in even ones, the lower envelope. This judgment is also confirmed by the graphs in fig. 2, which presents the results of the change in speed over time for sections $x = 24,\ 600, 912$ и $1176\ м$. Here one can observe a gradual transition to a regime with a constant speed $5\ м/с$.

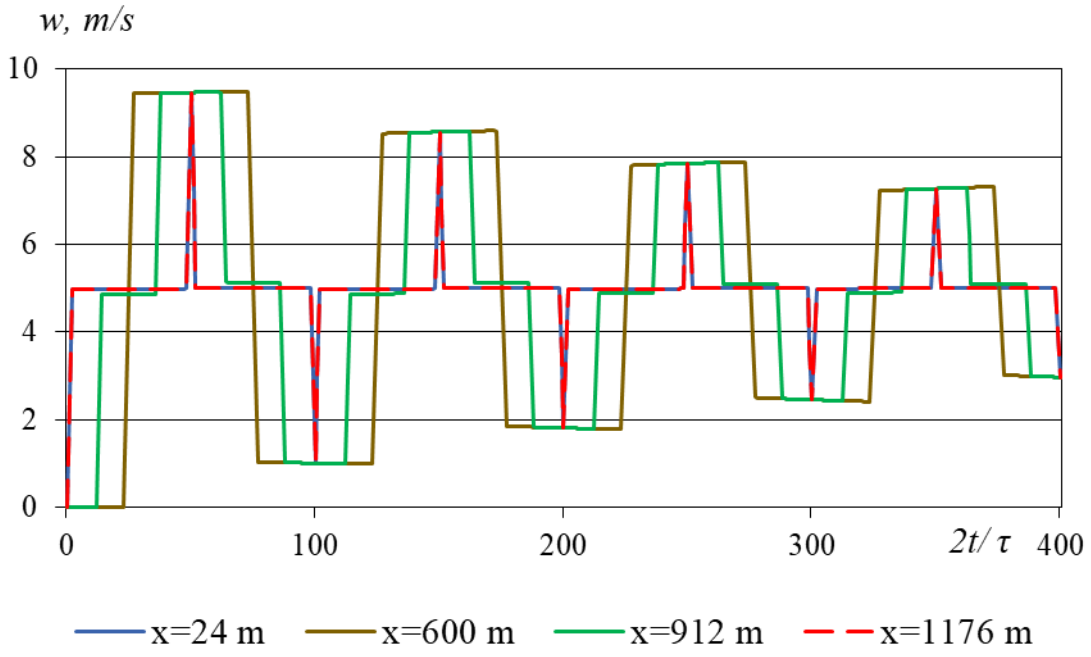

Fig. 2. Flow velocity graphs in different sections of the site.

For data see fig. 1

Fragments of the pressure dynamics for this case are shown in fig. 3 and 4. The following features of these graphs can be highlighted.

At the same time, the pressure has the highest and lowest values, related to the waves of compaction and resolution.

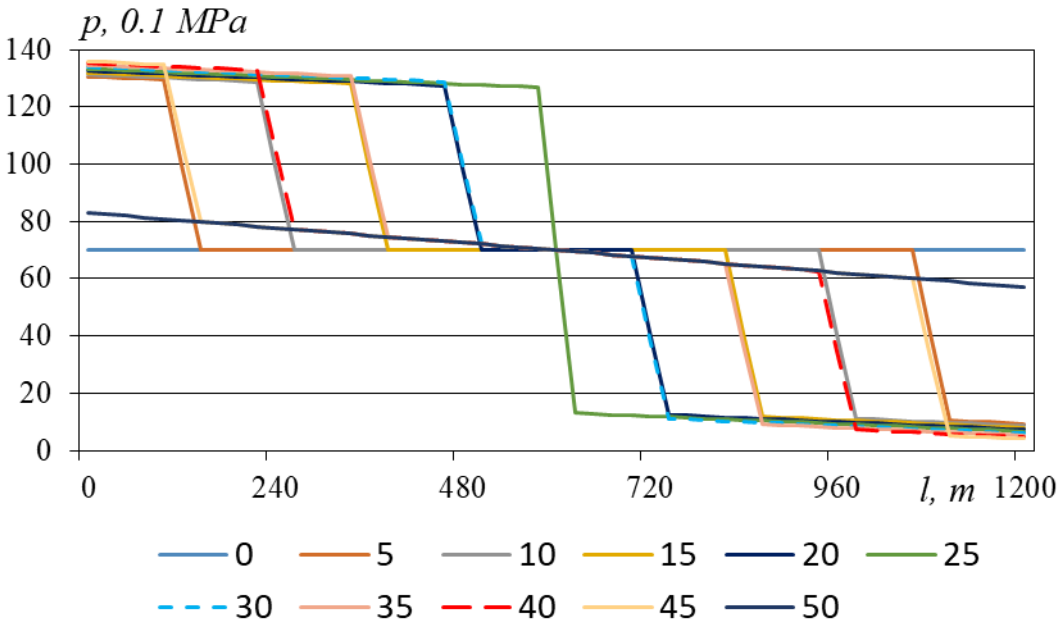

Fig. 3. Pressure curves in the first conditional period of the process of wave propagation and reflection. Data see fig. 1

The envelopes of the pressure curves gradually rotate around the point $(600 м, 7.0 МПа)$, gradually passes to a state that corresponds to a constant speed $5\ м/с$.

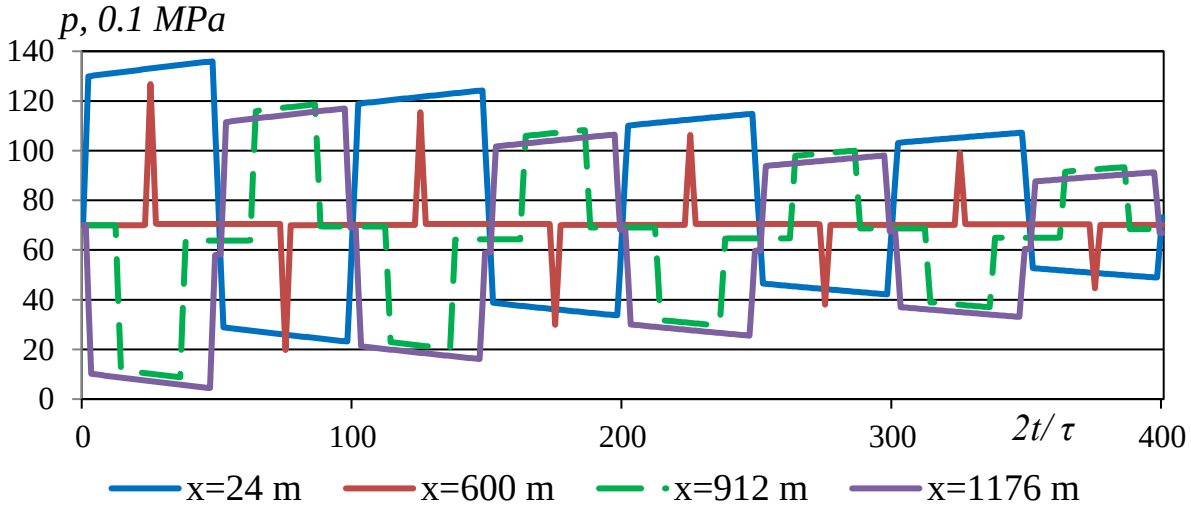

Fig. 4. Change in pressure over time in different sections of the pipeline. Data see fig. 1

The increase in pressure in the upper steps and the decrease in pressure in the lower steps in the case under consideration are determined by the Hugoniot relation , which assumes the constancy of the total pressure – the sum of the hydrostatic pressure and the dynamic pressure.

Conclusion

A mathematical model of the dynamic behavior of an elementary section of a pipeline with relief is proposed, based on quasi-one-dimensional equations of momentum and mass transfer of liquid. It is shown that the use of total pressure allows the equations to be reduced to the standard form of equations for pipeline transport on horizontal sections.

To solve the problem where the laws of flow velocity change over time are given at the ends of the section, a separate equation for velocity was composed, which has a telegraph type. This equation was solved by the Fourier method. The eigenfunctions of the problem with homogeneous boundary conditions are the sinusoidal function $X_n(x) = \sin\frac{\pi n}{l}x$ and three types of amplitudes describing the decaying disturbances:

$$Y_n(t) = \begin{cases} e^{-at}\left(A_n ch\sqrt{D_n}t + B_n sh\sqrt{D_n}t\right) \text{ при } D_n > 0, \\ e^{-at}\left(A_n + B_n t\right) \text{ при } D_n = 0, \\ e^{-at}\left(A_n \cos\sqrt{|D_n|}t + B_n \sin\sqrt{|D_n|}t\right) \text{ при } D_n < 0. \end{cases}$$

It is established that high-frequency components of disturbances decay exponentially (first group of solutions), while low-frequency oscillations (third group) persist longer. In the case of quasi-resonant frequency (second group), the amplitude of disturbances contains a linearly growing factor, but exponential decay compensates for this growth, preventing the development of stable resonance.

The results of the study were obtained for a developed turbulent flow $(\lambda = const)$ in a pipeline with a circular cross-section. To apply these results in calculations of pipelines with other cross-section shapes, the double equivalent radius of the pipeline can be used instead of the diameter D.

Literatures

1. Lewandowski A. New Numerical Methods For Transient Modeling of Gas Pipeline Networks. – New Mexico: Pipeline Simulation Interest Group, 1995.
2. Charny I.A. Unsteady motion of real fluid in pipes. 2nd ed. – M.: Nedra, 1975. – 296 p.,
3. Seleznev V.E., Aleshin V.V., Pryalov S.N. Modern computer simulators in pipeline transport. Mathematical methods of modeling and practical application / Ed. V.E.Seleznev . – M.: MAKS Press, 2007. – 200 p.
4. Charny , Korotaev Yu.P., Shirkovsky A.I. Extraction, transport and underground storage of gas. – M.: Nedra, 1997. – 487 p.
5. Fayzullaev D.F., Mukuk K.V. Sequential movements of liquids. – Tashkent: Fan, 1976. – 184 p.
6. Grachev V.V., Shcherbakov S.G., Yakovlev E.I. Dynamics of pipeline systems. – M.: Nauka, 1987. – 438 p.
7. Grachev V.V., Huseynzade M.A., Ksendz B.I., Yakovlev E.I. Complex pipeline systems. – M.: Nedra, 1982. – 256 p.
8. Atena A., Tekalign W., Muche T. Steady State Gas Flow in Pipeline Networks: Existence and Uniqueness of Solution // Journal of Applied Mathematics and Physics, (2020) 8 , 1155-1167. doi : 10.4236/jamp.2020.86087
9. Khujaev I.K., Bozorov O.Sh. , Mamadaliev Kh.A. , Aminov H.Kh. , Akhmadjonov SS Finite-difference method for solving nonlinear equations of traveling waves in main gas pipelines // Problems computing And applied mathematics . – Tashkent , 2020, No. 5(29). – P. 95-107 .

10. Rajabov MIND,. Zhonkobilov U.U. Calculation of hydraulic shock absorbers in long pressure pipelines of pumping stations. - Tashkent: Intellect nashriyot . 2022. - 130 p.
11. Budak B.M., Samarskii A.A., Tikhonov A.N. Collection of problems in mathematical physics. - M.: Nauka, 1972. - 678 p., Charny
12. Tikhonov A.N., Samarsky A.A. Equations of mathematical physics. – M.: Nauka, 1977. – 736 p., Budak
13. Khujaev I.K., Khojikulov Sh.Sh., Shamsutdinova N.Sh. The Sturm- Liouville problem for an arbitrary combination of inhomogeneous boundary conditions of the first, second and third kinds of boundary conditions / / Problems of Computational and Applied Mathematics, 2023, No. 2 (47). - P. 125-128.
14. Kamke E. Handbook of ordinary differential equations. – Moscow: Nauka, 1976. – 576 p.
15. Bronstein I.N., Semendyaev K.A. Handbook of Mathematics. – M.: Nauka, 1964. – 608 p.