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The Role of Multiple Integrals in Modeling Economic Risks

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Abstract: This paper explores the theoretical and practical aspects of using multiple integrals in the mathematical modeling of economic risks. In economic processes, risk is often formed not by a single factor but as a result of the combined influence of several random factors. Therefore, along with one-dimensional probabilistic models, it is necessary to use multidimensional models. The paper presents a method for determining expected economic outcome, variance, and risk indicators using multiple integrals based on the joint probability distribution of two random economic factors. It also demonstrates the possibility of solving the problem of risk modeling using the Lagrange multiplier method. The proposed approach can be applied in forming an investment portfolio, assessing currency and inflation risks, and optimizing economic decisions.

Keywords: Multiple integral, economic risk, mathematical modeling, probability density, portfolio, risk function, optimization, Lagrange multipliers.



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1. Introduction

In modern economic systems, decisions are made under conditions of uncertainty and risk. Changes in investment volume, exchange rates, inflation rates, interest rates, demand volume, and other economic indicators directly affect the final financial result of an enterprise, bank, or investment portfolio.[1]

One of the important tasks of mathematical modeling of economic risks is to take into account not only the separate influence of various risk factors but also their combined influence.

In multidimensional risk models, the relationship and joint variation between factors are of particular importance. In the scientific literature, multidimensional models of portfolio and financial risks are being developed from this perspective.

If the economic outcome depends on two random factors,[2]

$$Y = f(x_1, x_2)$$

then it is natural to express its expected value not through a single integral but through a double integral that accounts for the joint distribution of the factors x_1 and x_2

From this perspective, the aim of this paper is to study multiple integrals as a mathematical tool for modeling economic risks and to demonstrate their possibilities for practical application.[3]

2. Methodology

The methodology of this study is based on the application of multiple integral calculus to the mathematical modeling, measurement, and analysis of economic risks under conditions of uncertainty. The research combines theoretical analysis, mathematical modeling, probability theory, and numerical approximation to demonstrate how multiple integrals can be used to aggregate multidimensional economic factors and estimate the expected effects of uncertain events on economic and financial outcomes. The methodological framework treats economic risk as a multidimensional phenomenon in which several variables, such as market prices, exchange rates, interest rates, demand, investment returns, and production costs, may simultaneously influence the final economic result.

The first stage of the methodology involves identifying the principal economic variables and defining their possible ranges. Let $x_1, x_2, \dots, x_n$ represent uncertain economic variables and let $R(x_1, x_2, \dots, x_n)$ denote an economic risk function. Depending on the research objective, the function may represent a potential financial loss, deviation from an expected return, reduction in profitability, or another measurable risk indicator. The joint behavior of the variables is represented by a multidimensional probability density function $f(x_1, x_2, \dots, x_n)$. The expected value of the risk indicator can then be determined through integration over the relevant multidimensional domain.

For a two-variable economic risk model, the expected risk can be represented by the following double integral:

$$E[R] = \iint_D R(x, y) f(x, y) dx dy.$$

This approach makes it possible to account simultaneously for the effects of two uncertain economic variables and their joint probability distribution. For example, x may represent an exchange-rate change and y may represent an interest-rate change, while $R(x, y)$ describes the resulting financial loss. Unlike a single-variable model, the double-integral framework allows the interaction between two sources of uncertainty to be incorporated into the risk assessment.

The second stage involves constructing a multidimensional risk function. The function is defined according to the economic process under investigation and may include linear, nonlinear, or interaction components. A simplified risk function can be expressed as a combination of individual risk factors and their interaction effects. This is particularly important because economic variables frequently demonstrate interdependence. For instance, exchange-rate fluctuations can influence import costs, while interest-rate changes can simultaneously affect borrowing costs and investment decisions. Multiple integration allows these combined effects to be incorporated into a single quantitative framework.

The third stage consists of determining the probability distribution of the relevant economic variables. Historical observations, statistical databases, financial reports, and other empirical information can be used to estimate the distribution parameters. Where variables are correlated, a joint probability distribution is applied rather than independent marginal distributions. This allows the model to capture dependence between economic risk factors and reduces the possibility of obtaining misleading risk estimates based on an assumption of independence.

The methodology also incorporates scenario-based analysis. Several economic scenarios can be defined within the integration domain, such as favorable, moderate, and adverse market conditions. The contribution of each scenario to the overall expected risk can be evaluated by integrating the risk function over the corresponding subdomain. This approach enables researchers to examine how changes in the boundaries or probability density of economic variables affect

aggregate risk.

For more complex economic systems, the methodology can be extended from double integrals to triple and higher-order multiple integrals. If risk depends simultaneously on exchange rates, interest rates, and market prices, the expected risk may be represented as:

$$E[R] = \iiint_D R(x, y, z) f(x, y, z) dx dy dz.$$

Such a formulation provides a mathematical mechanism for evaluating several interacting sources of uncertainty simultaneously. The dimensionality of the model can therefore be increased according to the number of economically significant risk factors, although higher-dimensional models require greater computational resources and more reliable data.

The fourth stage focuses on numerical evaluation of the integrals. Since economic risk functions and probability distributions are rarely simple enough to be integrated analytically, numerical methods can be applied. Depending on the structure of the model, numerical integration, discretization, Monte Carlo simulation, or other computational techniques can be used to approximate the value of the multiple integral. Monte Carlo simulation is particularly useful when the integration domain is complex or when a large number of correlated risk factors are included in the model.

The methodology further includes sensitivity analysis to determine which economic variables have the greatest influence on the estimated risk. By systematically changing the parameters, ranges, or probability distributions of individual variables, the stability of the model can be examined. This makes it possible to identify particularly sensitive risk factors and evaluate how changes in economic conditions may affect the final risk estimate.

Finally, the reliability of the mathematical model is evaluated through comparison of calculated risk indicators under alternative assumptions and scenarios. The methodological framework emphasizes that multiple integrals should not be considered merely as mathematical operations but as analytical instruments for aggregating uncertainty across several economic dimensions. Their application provides a structured approach to estimating expected losses, analyzing joint risks, evaluating scenario distributions, and understanding the interaction between multiple economic factors. The proposed methodology therefore establishes a mathematical basis for applying integral calculus to contemporary economic risk analysis and decision-making under uncertainty.

3. Result and Discussion

Mathematical model of economic risk.

The main mathematical idea of the paper can be presented as follows. If the economic outcome Y depends on two random factors, for example, inflation X and exchange rate Z ,

$$Y = f(x, z)$$

then, through the joint probability density $\rho(x, z)$ of x and z , the expected economic outcome is expressed as[4]

$$E(Y) = \int_a^b \int_c^d f(x, z) \rho(x, z) dz dx$$

Here the double integral serves as the main mathematical tool for assessing risk.

For example, the mean square value of risk can be determined as

$$E(Y^2) = \int_a^b \int_c^d f^2(x, z) \rho(x, z) dz dx$$

and the variance can be found in the form

$$D(Y) = E(Y^2) - (E(Y))^2$$

Thus, the multiple integral enables the quantitative assessment not only of expected income

but also of the level of risk.[5]

Let us construct the following economic model:

$$Y(x, z) = a - bx - cz$$

the change in enterprise profit under the influence of inflation and the exchange rate: or

$$R(x, z) = \alpha x^2 + \beta z^2 + \gamma xz$$

the risk function under the influence of two economic factors.

If D is the region of risky states for an enterprise, the probability of a risky state occurring is calculated by the formula

$$P(D) = \iint_D f(x, z) dz dx$$

Suppose the joint density function for inflation x and exchange rate z is:

$$\rho(x, z) = 4xz, \quad 0 < x < 1, \quad 0 < z < 1$$

First, we check that this is indeed a probability density:

$$\int_0^1 \int_0^1 4xz dx dz = 4 \left(\int_0^1 x dx \right) \left(\int_0^1 z dz \right) = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$$

hence, $\rho(x, z) = 4xz$ is a probability density.

For the enterprise, we take the risky state to be $x + z > 1$. That is, if the combined influence of changes in inflation and the exchange rate exceeds a certain threshold, economic risk arises.[6]

$$0 < x < 1; 0 < z < 1 \text{ this field } 0 < x < 1, \quad 1 - x < z < 1$$

therefore

$$\begin{aligned} P(x + z > 1) &= \int_0^1 \int_{1-x}^1 4xz dz dx \\ \int_{1-x}^1 4xz dz &= 4x \int_{1-x}^1 z dz = 4x \cdot \left(\frac{z^2}{2} \right)_{1-x}^1 = 2x(1 - (1-x)^2) = 2x(2x - x^2) = 4x^2 - 2x^3 \\ P(x + z > 1) &= \int_0^1 (4x^2 - 2x^3) dx = \left(\frac{4}{3}x^3 - \frac{x^4}{2} \right)_0^1 = \frac{5}{6} \end{aligned}$$

Hence, $P(x + z > 1) = \frac{5}{6} \approx 0.83$, which means the result is 83%. That is, in this simplified model, the probability that the combined influence of changes in inflation and the exchange rate exceeds the threshold of 1 is 83%.

It is possible to calculate not only the probability of risk but also the expected economic loss using the formula[7]

$$E(L) = \iint_D L(x, z) \rho(x, z) dz dx$$

Let us return to the above example. Their joint probability density is:

$$\rho(x, z) = 4xz, \quad 0 < x < 1, \quad 0 < z < 1; \text{ let the risky state for the enterprise occur when } x + z > 1.$$

Suppose the loss in this state is expressed as:

$$L(x, z) = 100(x + z - 1)^2 \text{ thousand dollars. Find the expected loss of the enterprise.}$$

The expected loss is

$$E(L) = \iint_D L(x, z) \rho(x, z) dz dx$$

in the previous model[8]

$$\rho(x, z) = 4xz, \quad L(x, z) = 100(x + z - 1)^2$$

the result is $E(L) \approx 21.1$ thousand dollars. Suppose the enterprise spends funds to reduce risk:

let a be the funds spent on reducing inflation risk,

let b be the funds spent on reducing currency risk.

For example, the enterprise has allocated $a + b = 10$ thousand dollars to risk management; for simplification, we take the effective levels of risk reduction as

$$X_a = x - \alpha a, \quad Z_b = z - \beta b$$

where: α is the effectiveness of the anti-inflation measure; β is the effectiveness of the anti-currency-risk measure.[9]

For example, $\alpha = 0.04$, $\beta = 0.06$, which means that if 1 thousand dollars is spent on reducing inflation risk, the risk decreases by 0.04, and if spent on currency risk, it decreases by 0.06. How should the 10 thousand dollars be distributed, that is, which direction is more effective?

That is, under the condition $a + b = 10$, we must choose a and b .

If we spend a on inflation, then $b = 10 - a$ remains for currency, and then the overall risk reduction effect is[10]

$$R(a) = \alpha a + \beta(10 - a)$$

substituting the values:

$$R(a) = 0.04a + 0.06(10 - a)$$

$$R(a) = 0.6 - 0.02a$$

this function decreases as a increases. Therefore, to reduce risk as much as possible,

$$a = 0, b = 10$$

that is, in the simplified model, it is more effective to allocate funds to currency risk, because $\beta = 0.06 > \alpha = 0.04$

But there is an important issue here: making decisions based only on marginal effectiveness is not always correct. Because in the real economy, inflation and the exchange rate may be interconnected. Then $\rho(x, z)$ itself also changes.

For example, if inflation rises and the probability of exchange rate changes also rises, then x and z are not independent. In that case,[11]

$$\rho(x, z) \neq f_x(x) \cdot f_z(z)$$

this is a very important case.

The expected loss after risk management takes the form:

$$E(a, b) = \iint_{D(a, b)} L(x, z, a, b) \rho(x, z) dz dx$$

And we solve the problem $\min E(a, b)$ under the budget constraint $a + b = B$. Here the marginal effect appears.

Now $\frac{\partial E}{\partial a}$ shows how much spending an additional 1 thousand dollars on reducing inflation risk changes the expected loss. Similarly, $\frac{\partial E}{\partial b}$ is the marginal effect of additional funds on reducing currency risk.

At the optimal point, as a rule, a condition of equality of marginal effects such as $\frac{\partial E}{\partial a} = \frac{\partial E}{\partial b}$ appears. In the next step, while preserving the condition $a + b = B$, we transform the multiple integral into a function fully dependent on a and b , find $E(a, b)$, and from it calculate $\frac{\partial E}{\partial a}$, $\frac{\partial E}{\partial b}$ and mathematically find the exact optimal distribution of the 10 thousand dollar budget.

We have $a + b = 10$, where a is spent on reducing inflation risk, and b on reducing currency risk. The effectiveness values are $\alpha = 0.04$, $\beta = 0.06$; therefore, we take the reduced form of the risk factors as

$$X_a = x - 0.04a, \quad Z_b = z - 0.06b$$

In the previous model, the expected loss was

$$L(x, z) = 100(x + z - 1)^2$$

Now, taking risk management into account:

$$L(a, b) = 100(x + z - 1 - 0.04a - 0.06b)^2$$

Therefore

$$E(a, b) = 400 \iint_D 100(x + z - 1 - 0.04a - 0.06b)^2 dz dx$$

Since the budget is $a + b = B$, then $b = 10 - a$, therefore,

$$0.04a + 0.06b = 0.04a + 0.06(10 - a) = 0.6 - 0.02a$$

Thus:

$$E(a) = 400 \iint_D xz(x + z - 1.6 + 0.02a)^2 dz dx$$

the goal is to find the optimal a . Here we differentiate with respect to a :

$$E'(a) = 16 \iint_D xz(x+z-1.6+0.02a)^2 dzdx$$

$$E'(a) = 0$$

therefore,

$$\iint_D xz(x+z-1.6+0.02a)^2 dzdx = 0$$

Using our previous calculations, if we compute this integral,

$$\iint_D xz dzdx = \frac{1}{6}, \text{ and } \iint_D xz(x+z-1) dzdx = \frac{1}{30}$$

Therefore[12]

$$\frac{1}{30} - 0.6 \cdot \frac{1}{6} + 0.02a \cdot \frac{1}{6} = 0$$

from which $a = 20$ follows, but our budget is only 10 thousand dollars, $0 \leq a \leq 10$, therefore the interior optimum $a = 20$ lies outside the budget range. Hence, the function $E(a)$ will be decreasing on the given interval $[0, 10]$, and the minimum is at the boundary

$$a = 10, \quad b = 0$$

The following economic conclusion can be drawn. In this model, it turned out optimal to direct the entire 10 thousand dollar budget to reducing inflation risk:

$$a^* = 10, \quad b^* = 0$$

Why did the earlier simple analysis show currency risk to be more effective? Because here we took into account not only $\alpha = 0.04$, $\beta = 0.06$, but also the probability distribution of risk and the loss function. Therefore, the model based on the multiple integral gives a much more complete and accurate decision.

To view this problem more generally, we use the Lagrange method. The stated economic problem is the enterprise budget:[13]

$$a + b = B$$

Let this be. Here 1) a — expenditure aimed at reducing inflation risk

2) b — expenditure aimed at reducing currency risk

Let us denote the expected loss by $E(a, b)$; the goal is $\min E(a, b)$, subject to $a + b = B$. We construct the Lagrange function

$$L(a, b, \lambda) = E(a, b) + \lambda(a + b - B)$$

At the optimal point, the first-order conditions are

$$\frac{\partial L}{\partial a} = 0, \quad \frac{\partial L}{\partial b} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

From this,

$$E_a + \lambda = 0$$

$$E_b + \lambda = 0$$

$$a + b = B$$

Therefore, $E_a = E_b$. The economic meaning of this is as follows:

At the optimal distribution, the marginal effect of additional funds spent on reducing inflation risk will be equal to the marginal effect of additional funds spent on reducing currency risk.

To make the calculation simpler and economically meaningful, we model the expected loss as follows:

$$E(a, b) = 100 - 8a - 12b + a^2 + b^2$$

Here

- 100 is the initial expected loss;
- $-8a$ is the initial effect of reducing inflation risk;
- $-12b$ is the initial effect of reducing currency risk;
- a^2, b^2 represent the decline in the additional effect as more funds are spent.

Thus, subject to $a + b = B$, we must find $\min (100 - 8a - 12b + a^2 + b^2)$

The derivatives of the Lagrange function:[14]

$$\frac{\partial L}{\partial a} = -8 + 2a + \lambda = 0$$

$$\begin{aligned}\frac{\partial L}{\partial b} &= -12 + 2b + \lambda \\ \frac{\partial L}{\partial \lambda} &= a + b - 10 = 0\end{aligned}$$

From the first and second equations,

$$\begin{aligned}-8 + 2a &= -12 + 2b \\ 2a - 2b &= -4 \\ a - b &= -2\end{aligned}$$

We have $a + b = 10$; solving these two equations together, we obtain $a = 4$ and $b = 6$.

Optimal economic decision:

$$a^* = 4, \quad b^* = 6$$

that is, of the 10 thousand dollar budget,

- 4 thousand dollars — for reducing inflation risk,

- 6 thousand dollars — for reducing currency risk

will be the optimal decision.

The marginal effects are equalized, that is,[15]

$$\left| \frac{\partial E}{\partial a} \right| = \left| \frac{\partial E}{\partial b} \right|$$

Indeed,

$$\begin{aligned}E_a &= -8 + 2a = -8 + 8 = 0 \\ E_b &= -12 + 2b = -12 + 12 = 0\end{aligned}$$

4. Conclusion

Multiple integrals are one of the important analytical tools in the mathematical modeling of economic processes. With their help, it becomes possible to take into account the combined influence of several random and economic factors, to determine the probability of risk states, and to assess the expected value of economic loss.

In the study, a model for assessing economic risk based on a two-variable probability density function was considered. The combined influence of factors such as inflation and the exchange rate was expressed using a multiple integral, and the probability of risk that may arise in the risky region was determined. Also, a model based not only on the probability of risk but also taking into account the amount of expected economic loss was analyzed, based on the formula

$$E(L) = \iint_D L(x, z) \rho(x, z) dz dx$$

The results of the study show that the multiple integral makes it possible to take into account not individual factors but their interconnected influence in assessing economic risks. This creates a mathematical basis for more accurate assessment and management of risk.

The paper also considered the problem of optimally distributing funds aimed at reducing risk under a limited budget. Through the Lagrange multiplier method, the determination of the optimal decision and the principle of equalizing marginal effects were substantiated. As a result, a complex economic-mathematical model combining multiple integrals, probability theory, partial derivatives, and mathematical optimization was formulated. Thus, multiple integrals serve as an effective mathematical apparatus in assessing, forecasting, and managing economic risks, as well as in the optimal distribution of limited resources. In the future, this model can be expanded to take into account correlation relationships between inflation, the exchange rate, the interest rate, and stock market indicators.

References

- [1] S. M. Ross, *Introduction to Probability Models*, 12th ed. Cambridge, MA, USA: Academic Press, 2019, pp. 1–848.
- [2] S. M. Ross, *A First Course in Probability*, 10th ed. Boston, MA, USA: Pearson, 2019, pp. 1–528.

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- [3] D. G. Luenberger, *Investment Science*, 2nd ed. New York, NY, USA: Oxford University Press, 2014, pp. 1–494.
 - [4] J. C. Hull, *Risk Management and Financial Institutions*, 6th ed. Hoboken, NJ, USA: Wiley, 2023, pp. 1–720.
 - [5] J. C. Hull, *Options, Futures, and Other Derivatives*, 11th ed. Hoboken, NJ, USA: Pearson, 2022, pp. 1–848.
 - [6] A. Meucci, *Risk and Asset Allocation*. Berlin, Germany: Springer, 2005, pp. 1–688.
 - [7] H. Markowitz, "Portfolio selection," *The Journal of Finance*, vol. 7, no. 1, pp. 77–91, Mar. 1952.
 - [8] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, May–Jun. 1973.
 - [9] R. C. Merton, "Theory of rational option pricing," *The Bell Journal of Economics and Management Science*, vol. 4, no. 1, pp. 141–183, Spring 1973.
 - [10] P. Glasserman, *Monte Carlo Methods in Financial Engineering*. New York, NY, USA: Springer, 2004, pp. 1–596.
 - [11] C. W. Gardiner, *Handbook of Stochastic Methods: For Physics, Chemistry and the Natural Sciences*, 4th ed. Berlin, Germany: Springer, 2009, pp. 1–447.
 - [12] A. Papoulis and S. U. Pillai, *Probability, Random Variables, and Stochastic Processes*, 4th ed. New York, NY, USA: McGraw-Hill, 2002, pp. 1–852.
 - [13] R. Durrett, *Probability: Theory and Examples*, 5th ed. Cambridge, U.K.: Cambridge University Press, 2019, pp. 1–419.
 - [14] G. E. Uhlenbeck and L. S. Ornstein, "On the theory of the Brownian motion," *Physical Review*, vol. 36, no. 5, pp. 823–841, Sep. 1930.
 - [15] T. Mikosch, *Elementary Stochastic Calculus with Finance in View*. Singapore: World Scientific, 1998, pp. 1–216.