

Article

Laplace Equation and Its Form in Polar and Spherical Coordinate Systems

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Abstract: This article examines the forms of the Laplace equation in polar and spherical coordinate systems. The study derives the transformation formulas from Cartesian coordinates to polar and spherical coordinates, as well as the corresponding differential elements and expressions of the Laplace operator in these coordinate systems. The obtained results are of significant importance in solving problems related to mathematical physics and differential equations.

Keywords: Laplace equation, Poisson equation, polar coordinates, spherical coordinates, Laplace operator, harmonic function, mathematical physics, differential equations, orthogonal coordinates, coordinate transformation.

1. Introduction

Problems of studying stationary, i.e., time-invariant, physical processes are presented based on elliptic-type equations.

In this book, we consider the Laplace equation, the simplest representative of elliptic equations[1]:

$$\Delta u = \sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2} = 0$$

We limit ourselves to the study of [1]. The methods developed for this equation (of course, with certain additional assumptions) also apply to the study of more general elliptic equations[2].

We wrote the Laplace equation for arbitrary $m \geq 2$, since for $m \geq 3$ the concepts and formulas are expressed identically, whereas for $m = 2$ they are somewhat different. Therefore, we will study equation (1) for $m \geq 3$ cases and, where necessary, explain the difference for $m = 2$. Henceforth, when we refer to a point x in an m -dimensional space, we mean the point $(x=x_1, x_2, \dots, x_m)$ with coordinates $x_1, x_2, \dots, x_m$ [3-5].

If equation (1)

$$\Delta u = F(x) \quad [2]$$

In its appearance, it is called the nonlinear Laplace equation or Poisson equation.

When $m = 2$, that is, the equations (1) and (2) in the (x, y) plane

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad [3]$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad [4]$$

It will appear as follows.

Let us consider an arbitrary closed surface G (it need not be connected). Let this surface be the boundary of a finite (Figure 10) or infinite (Figure 11) domain D [6-9].

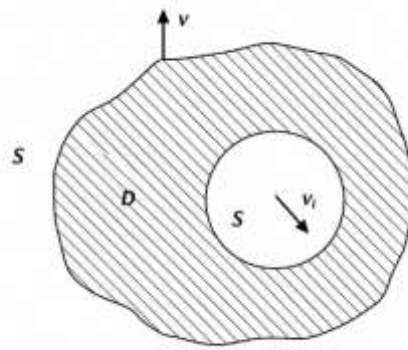


Figure 1. Movement of an Inclusion within a Deforming Material Matrix

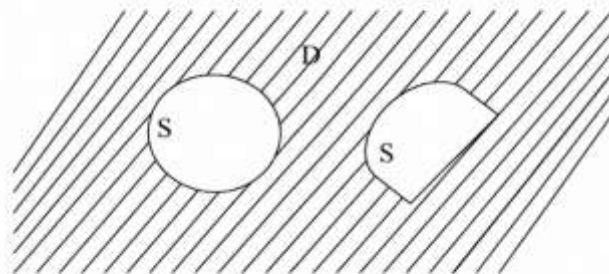


Figure 2. Geometric Configurations of an Inclusion Embedded in a Material Matrix

In such bounded or unbounded domains, we study the properties of the solutions of the Laplace equation [1].

Definition: 1) In a bounded domain D , a function $u(x,y)$ that is twice continuously differentiable with respect to both variables and satisfies (1) the Laplace equation is called a harmonic function[10].

2) A function $u(x)$ is called harmonic in the unbounded domain D if it has two continuous derivatives at every point within a finite distance of the origin and satisfies the Laplace equation, and in the infinity as $O\left(\frac{1}{|x|^{m-2}}\right)$ tends to zero[11]. This latter condition means that for sufficiently large $|x| \dots$

$$|u(x)| \leq \frac{C}{|x|^{m-2}} \quad [5]$$

means let m be the dimension of the space, and C is some constant.

$|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_m^2}$. A function that is harmonic in a two-dimensional infinite domain must, by (5), be bounded at infinity.

In many physical and mathematical problems, especially in regions with rotational or radial symmetry, using the Cartesian coordinate system can be inconvenient. In such cases, the polar coordinate system makes it possible to simplify the problem[12].

2. Materials and Methods

This study employed theoretical and analytical research methods to investigate the forms of the Laplace equation in polar and spherical coordinate systems. The mathematical analysis method was used to derive the Laplace operator from Cartesian coordinates into polar and spherical coordinates. The coordinate transformation method

was applied to establish relationships between different coordinate systems and to obtain the corresponding differential elements. In addition, the comparative method was utilized to compare the representations of the Laplace equation in Cartesian, polar, and spherical coordinates. The obtained results were generalized through the deductive and synthesis methods, providing a theoretical basis for solving problems in mathematical physics and differential equations.

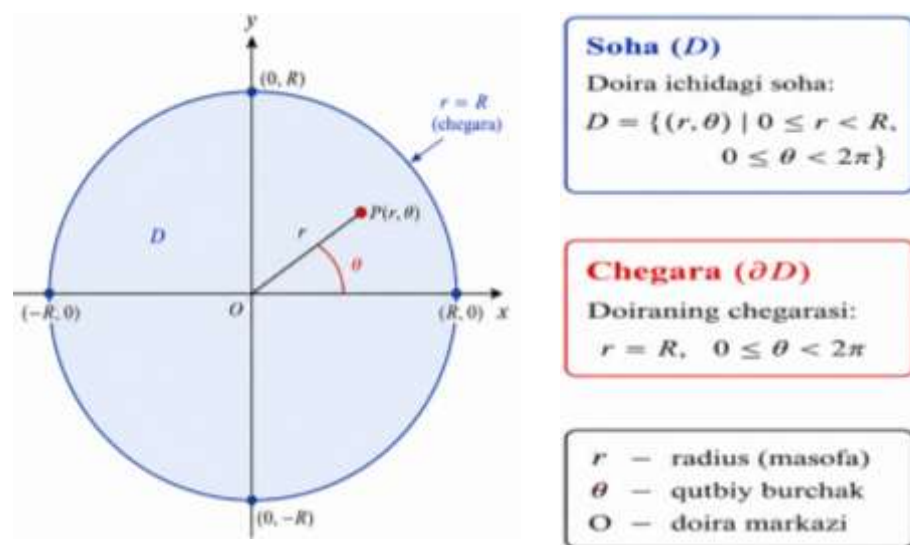
3. Results and Discussion

The polar coordinate system is a coordinate system that represents points in a plane by a radius and an angle, and it is especially widely used in problems involving symmetry.

Transformation from Cartesian coordinates to polar coordinates[13]:

A point $M(x, y)$ in the plane is written in polar coordinates as $M(r, \theta)$. Here r is the distance of the point from the origin, and θ is the angle it makes with the Ox axis[14].

From the following figure we derive the following equations[15]:



$$\cos \theta = \frac{x}{r} \quad \sin \theta = \frac{y}{r} \quad x = r \cos \theta, \quad y = r \sin \theta \quad [6]$$

Now we find r ; to do this, we square and add x and y .

$$x^2 + y^2 = (r \cos \theta)^2 + (r \sin \theta)^2 \quad [7]$$

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$x^2 + y^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$\cos^2 \theta + \sin^2 \theta = 1$ From the fact that (7), we obtain the following equations (8) and (9).

$$x^2 + y^2 = r^2 \quad [8]$$

$$r = \sqrt{x^2 + y^2} \quad [9]$$

From the corner, we obtain the following equalities:

$\tan \theta = \frac{y}{x}$ From this, we find the following $\theta = \arctan \frac{y}{x}$ [10]. We obtain the relationship between polar coordinates and Cartesian coordinates [6], [8], [9], [10], which is a combination of the equations.

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}$$

Now we begin the proof of the differential element. To do this, we differentiate the expressions in (6) with respect to the polar coordinates and obtain the following identity [11].

$$\begin{cases} dx = \cos \theta dr - r \sin \theta d\theta \\ dy = \sin \theta dr + r \cos \theta d\theta \end{cases} \quad [11]$$

Length element on the plane in Cartesian coordinates

$$ds^2 = dx^2 + dy^2 \quad [12]$$

[12] Substituting dx and dy in their places, we obtain the following equations.

$$ds^2 = (\cos \theta dr - r \sin \theta d\theta)^2 + (\sin \theta dr + r \cos \theta d\theta)^2$$

$$ds^2 = \cos^2 \theta dr^2 - 2r \sin \theta \cos \theta dr d\theta + r^2 \sin^2 \theta d\theta^2 + \sin^2 \theta dr^2 + 2r \sin \theta \cos \theta dr d\theta + r^2 \cos^2 \theta d\theta^2$$

If we simplify the expression, we obtain the following equations. When we simplify, upon squaring [11] and then adding both sides of the equation, the system's quadratic terms cancel out. and we simplify by combining the same terms in the second-order differential into a single equation.

$$ds^2 = (\cos^2 \theta + \sin^2 \theta) dr^2 + r^2 (\sin^2 \theta + \cos^2 \theta) d\theta^2$$

By the trigonometric property, we obtain equation [13].

$$ds^2 = dr^2 + r^2 d\theta^2 \quad [13]$$

Now we obtain the surface element via the Jacobian and, as a result, we get the equality [14].

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

We will find the results one by one.

$$\frac{\delta x}{\delta r} = \cos \theta$$

$$\frac{\delta x}{\delta \theta} = -r \sin \theta \quad \frac{\delta y}{\delta r} = \sin \theta \quad \frac{\delta y}{\delta \theta} = r \cos \theta \quad [15]$$

Now we substitute the values of $n = 15$ into expression (14).

$$J = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

We calculate the determinant and find the Jacobian.

$$J = r \cos^2 \theta + r \sin^2 \theta$$

$$J = r(\cos^2 \theta + \sin^2 \theta) = r$$

Therefore, the surface element

$$dS = |J| dr d\theta$$

So

$$dS = r dr d\theta \quad [16]$$

After achieving the result, we now prove the first-order derivatives. If $u = u(x, y) = u(r, \theta)$, then by the chain rule it is as follows.

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta r} \frac{\delta r}{\delta x} + \frac{\delta u}{\delta \theta} \frac{\delta \theta}{\delta x} \quad [17]$$

[9] - we take the first-order derivative with respect to x and y from equality.

$$r = \sqrt{x^2 + y^2} \quad [9]$$

Now we will bring the following tenhliks.

$$\frac{\delta r}{\delta x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} \quad [18]$$

$$x = r \cos \theta \quad y = r \sin \theta \quad \frac{x}{r} = \cos \theta \quad \frac{\delta r}{\delta x} = \cos \theta \quad \tan \theta = \frac{y}{x}$$

$$\frac{\delta \theta}{\delta x} = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2} = -\frac{r \sin \theta}{r^2} = -\frac{\sin \theta}{r} \quad [19]$$

We form the equations. Taking equation [17] and substituting equations [18] and [19], we obtain the following equation.

$$\frac{\delta u}{\delta x} = \cos \theta \frac{\delta u}{\delta r} - \frac{\sin \theta}{r} \frac{\delta u}{\delta \theta} \quad [20]$$

Now we repeatedly do the same thing for y .

$$\frac{\delta u}{\delta y} = \frac{\delta u}{\delta r} \frac{\delta r}{\delta y} + \frac{\delta u}{\delta \theta} \frac{\delta \theta}{\delta y} \quad [21]$$

$$\frac{\delta r}{\delta y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r} = \frac{r \sin \theta}{r} = \sin \theta \quad [22]$$

$$\frac{\delta \theta}{\delta y} = \frac{x}{x^2 + y^2} = \frac{r \cos \theta}{r^2} = \frac{\cos \theta}{r} \quad [23]$$

$$\frac{\delta u}{\delta y} = \sin \theta \frac{\delta u}{\delta r} + \frac{\cos \theta}{r} \frac{\delta u}{\delta \theta} \quad [24]$$

We can combine [24] and [20] into a single equation as follows

$$u_x = \cos \theta \frac{\delta u}{\delta r} - \frac{\sin \theta}{r} \frac{\delta u}{\delta \theta}$$

[25]

$$u_y = \sin \theta \frac{\delta u}{\delta r} + \frac{\cos \theta}{r} \frac{\delta u}{\delta \theta}$$

Now we will prove the Laplace operator in polar coordinates:

$$\Delta u = u_{xx} + u_{yy}$$

[25] At equality, we have obtained the first-order derivatives; now we compute the second-order derivatives of these expressions by taking the first-order derivatives once more.

$$\begin{aligned} u_x &= \cos \theta \frac{\delta u}{\delta r} - \frac{\sin \theta}{r} \frac{\delta u}{\delta \theta} \frac{\delta}{\delta r} \left(\cos \theta u_r - \frac{\sin \theta}{r} u_\theta \right) = \cos \theta u_{rr} - \sin \theta \frac{\delta}{\delta r} \left(\frac{1}{r} u_\theta \right) \\ &= \cos \theta u_{rr} - \sin \theta \left(-\frac{1}{r^2} u_\theta + \frac{1}{r} u_{\theta r} \right) u_{xx} \\ &= \cos^2 \theta u_{rr} + \frac{\sin \theta \cos \theta}{r^2} u_\theta - \frac{\sin \theta \cos \theta}{r} u_{\theta r} + \frac{\sin^2 \theta}{r} u_r \\ &\quad - \frac{\sin \theta \cos \theta}{r} u_{r\theta} + \frac{\sin \theta \cos \theta}{r^2} u_\theta + \frac{\sin^2 \theta}{r} u_{\theta\theta} \\ u_{xx} &= (u_x)_x = \left(\cos \theta \frac{\delta u}{\delta r} - \frac{\sin \theta}{r} \frac{\delta u}{\delta \theta} \right)_x \end{aligned}$$

[26]

$$u_{yy} = (u_y)_y = \left(\sin \theta \frac{\delta u}{\delta r} + \frac{\cos \theta}{r} \frac{\delta u}{\delta \theta} \right)_y$$

We calculate them separately, and for x we substitute the following expression.

$$\begin{aligned} \frac{\delta}{\delta x} &= \cos \theta \frac{\delta}{\delta r} - \frac{\sin \theta}{r} \frac{\delta}{\delta \theta} \\ u_{xx} &= (u_x)_x = \left(\cos \theta \frac{\delta u}{\delta r} - \frac{\sin \theta}{r} \frac{\delta u}{\delta \theta} \right)_x = \frac{\delta}{\delta x} \left(\cos \theta \frac{\delta u}{\delta r} - \frac{\sin \theta}{r} \frac{\delta u}{\delta \theta} \right) \\ u_{xx} &= \left(\cos \theta \frac{\delta}{\delta r} - \frac{\sin \theta}{r} \frac{\delta}{\delta \theta} \right) \left(\cos \theta \frac{\delta u}{\delta r} - \frac{\sin \theta}{r} \frac{\delta u}{\delta \theta} \right) \\ u_{xx} &= \cos \theta \frac{\delta}{\delta r} \left(\cos \theta \frac{\delta u}{\delta r} - \frac{\sin \theta}{r} \frac{\delta u}{\delta \theta} \right) - \frac{\sin \theta}{r} \frac{\delta}{\delta \theta} \left(\cos \theta \frac{\delta u}{\delta r} - \frac{\sin \theta}{r} \frac{\delta u}{\delta \theta} \right) \\ u_{xx} &= \cos \theta \frac{\delta}{\delta r} \left(\cos \theta u_r - \frac{\sin \theta}{r} u_\theta \right) - \frac{\sin \theta}{r} \frac{\delta}{\delta \theta} \left(\cos \theta u_r - \frac{\sin \theta}{r} u_\theta \right) \quad (27) \end{aligned}$$

Here we take the derivative once with respect to r and θ .

$$\begin{aligned} \frac{\delta}{\delta r} \left(\cos \theta u_r - \frac{\sin \theta}{r} u_\theta \right) &= \cos \theta u_{rr} - \sin \theta \frac{\delta}{\delta r} \left(\frac{1}{r} u_\theta \right) = \cos \theta u_{rr} - \sin \theta \left(-\frac{1}{r^2} u_\theta + \frac{1}{r} u_{\theta r} \right) \\ \frac{\delta}{\delta \theta} \left(\cos \theta u_r - \frac{\sin \theta}{r} u_\theta \right) &= -\sin \theta u_r + \cos \theta u_{r\theta} - \frac{\cos \theta}{r} u_\theta - \frac{\sin \theta}{r} u_{\theta\theta} \end{aligned}$$

We now substitute these expressions into Equation [27].

$$\begin{aligned} u_{xx} &= \cos \theta \left(\cos \theta u_{rr} - \sin \theta \left(-\frac{1}{r^2} u_\theta + \frac{1}{r} u_{\theta r} \right) \right) - \frac{\sin \theta}{r} \left(-\sin \theta u_r + \cos \theta u_{r\theta} - \frac{\cos \theta}{r} u_\theta - \frac{\sin \theta}{r} u_{\theta\theta} \right) \\ u_{xx} &= \cos^2 \theta u_{rr} + \frac{\sin \theta \cos \theta}{r^2} u_\theta - \frac{\sin \theta \cos \theta}{r} u_{\theta r} + \frac{\sin^2 \theta}{r} u_r - \frac{\sin \theta \cos \theta}{r} u_{r\theta} + \frac{\sin \theta \cos \theta}{r^2} u_\theta + \frac{\sin^2 \theta}{r} u_{\theta\theta} \\ u_{xx} &= \cos^2 \theta u_{rr} - \frac{2 \sin \theta \cos \theta}{r} u_{\theta r} + \frac{\sin^2 \theta}{r} u_{\theta\theta} + \frac{\sin^2 \theta}{r} u_r + \frac{2 \sin \theta \cos \theta}{r^2} u_\theta \quad [28] \end{aligned}$$

Now we calculate the derivatives with respect to y.

$$\frac{\delta}{\delta y} = \sin \theta \frac{\delta}{\delta r} + \frac{\cos \theta}{r} \frac{\delta}{\delta \theta}$$

$$\begin{aligned} u_{yy} &= (u_y)_y = \left(\sin \theta \frac{\delta u}{\delta r} + \frac{\cos \theta}{r} \frac{\delta u}{\delta \theta} \right)_y = \frac{\delta}{\delta y} \left(\sin \theta \frac{\delta u}{\delta r} + \frac{\cos \theta}{r} \frac{\delta u}{\delta \theta} \right) \\ u_{yy} &= \left(\sin \theta \frac{\delta}{\delta r} + \frac{\cos \theta}{r} \frac{\delta}{\delta \theta} \right) \left(\sin \theta u_r + \frac{\cos \theta}{r} u_\theta \right) \quad [29] \end{aligned}$$

Here we take the derivative once with respect to r and θ .

$$\begin{aligned} \frac{\delta}{\delta r} \left(\sin \theta u_r + \frac{\cos \theta}{r} u_\theta \right) &= \sin \theta u_{rr} + \cos \theta \left(-\frac{1}{r^2} u_\theta + \frac{1}{r} u_{r\theta} \right) = \sin \theta u_{rr} - \cos \theta \frac{1}{r^2} u_\theta + \frac{\cos \theta}{r} u_{r\theta} \\ \frac{\delta}{\delta \theta} \left(\sin \theta u_r + \frac{\cos \theta}{r} u_\theta \right) &= \cos \theta u_r + \sin \theta u_{r\theta} - \frac{\sin \theta}{r} u_\theta + \frac{\cos \theta}{r} u_{\theta\theta} \frac{dy}{dx} \end{aligned}$$

We now substitute these expressions into equation (29).

$$\begin{aligned} u_{yy} &= \sin \theta \left(\sin \theta u_{rr} - \cos \theta \frac{1}{r^2} u_\theta + \frac{\cos \theta}{r} u_{r\theta} \right) + \frac{\cos \theta}{r} \left(\cos \theta u_r + \sin \theta u_{r\theta} - \frac{\sin \theta}{r} u_\theta + \frac{\cos \theta}{r} u_{\theta\theta} \right) \\ u_{yy} &= \sin^2 \theta u_{rr} - \frac{\sin \theta \cos \theta}{r^2} u_\theta + \frac{\sin \theta \cos \theta}{r} u_{r\theta} + \frac{\cos^2 \theta}{r} u_r + \frac{\sin \theta \cos \theta}{r} u_{r\theta} - \frac{\sin \theta \cos \theta}{r^2} u_\theta + \frac{\cos^2 \theta}{r^2} u_{\theta\theta} \end{aligned}$$

$$u_{yy} = \sin^2 \theta u_{rr} + \frac{2 \sin \theta \cos \theta}{r} u_{r\theta} + \frac{\cos^2 \theta}{r^2} u_{\theta\theta} - \frac{2 \sin \theta \cos \theta}{r^2} u_{\theta} + \frac{\cos^2 \theta}{r} u_r \quad [30]$$

[30] we form the equality. Now we generalize and combine the [28] and [30] equations, resulting in the following equality, and simplify using trigonometric identities.

$$u_{xx} = \cos^2 \theta u_{rr} - \frac{2 \sin \theta \cos \theta}{r} u_{r\theta} + \frac{\sin^2 \theta}{r^2} u_{\theta\theta} + \frac{\sin^2 \theta}{r} u_r + \frac{2 \sin \theta \cos \theta}{r^2} u_{\theta}$$

$$u_{yy} = \sin^2 \theta u_{rr} + \frac{2 \sin \theta \cos \theta}{r} u_{r\theta} + \frac{\cos^2 \theta}{r^2} u_{\theta\theta} - \frac{2 \sin \theta \cos \theta}{r^2} u_{\theta} + \frac{\cos^2 \theta}{r} u_r$$

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \quad [31]$$

[31] We obtain an equation. Now, according to the Laplace operator, we transform from the Cartesian coordinate system to the polar coordinate system

$$\Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad [32]$$

[32] expression of the Laplace equation in the polar coordinate system

Now we introduce the spherical coordinate system for the Laplace equation

The spherical coordinate system is a coordinate system based on representing points in three-dimensional space using a radius and two angles. This system is especially widely used in studying problems involving spheres, balls, or objects with central symmetry. In mathematical physics, particularly when solving Laplace's and Poisson's equations, spherical coordinates are very convenient[16].

A point $M(x,y,z)$ in three-dimensional space is expressed in spherical coordinates as $M(r,\theta,\phi)$. Here, r is the distance of the point from the origin

θ - polar angle

ϕ - azimuth angle

The transition from Cartesian coordinates to spherical coordinates is carried out using the following formulas[17].

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta$$

Conversely, spherical coordinates are defined as follows.

$$r = \sqrt{x^2 + y^2 + z^2} \quad \theta = \arccos \frac{z}{r} \quad \phi = \arctg \frac{y}{x}$$

In the spherical coordinate system

$r = \text{const}$ the surface of the sphere

$\theta = \text{const}$ the surface of the cone

$\phi = \text{const}$ Represents a semi-plane

Now we introduce the length element and the volume element in spherical coordinates.

$$ds^2 = dx^2 + dy^2 + dz^2 \quad [33]$$

[33] To express the length element in the three-dimensional Cartesian coordinate system, we project it onto the spherical coordinate system, introducing the following identities to obtain dx , dy , and dz . To do this, we take derivatives of the functions.

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi$$

$$\frac{\partial x}{\partial r} = \sin \theta \cos \phi$$

$$\frac{\partial x}{\partial \theta} = r \cos \theta \cos \phi$$

$$\frac{\partial x}{\partial \phi} = -r \sin \theta \sin \phi$$

$$dx = \sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi \quad [34]$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi$$

$$\frac{\partial y}{\partial r} = \sin \theta \sin \phi$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta \sin \phi$$

$$\frac{\partial y}{\partial \phi} = r \sin \theta \cos \phi$$

$$dy = \sin \theta \sin \phi dr + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi \quad [35]$$

$$dz = \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta$$

$$\frac{\partial z}{\partial r} = \cos \theta$$

$$\frac{\partial z}{\partial \theta} = -r \sin \theta$$

$$dz = \cos \theta dr - r \sin \theta d\theta \quad [36]$$

We now substitute the equations [34], [35], and [36] into the expression [33], yielding the following result.

$$dx^2 = (\sin \theta \cos \varphi dr + r \cos \theta \cos \varphi d\theta - r \sin \theta \sin \varphi d\varphi)^2$$

$$dy^2 = (\sin \theta \sin \varphi dr + r \cos \theta \sin \varphi d\theta + r \sin \theta \cos \varphi d\varphi)^2$$

$$dz^2 = (\cos \theta dr - r \sin \theta d\theta)^2$$

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad [37]$$

[37] The expression for the spherical representation of this length element now allows us to compute the volume element. We then determine the Laplace coefficient.

$$h_r = 1 \quad h_\theta = r \quad h_\varphi = r \sin \theta \quad [38]$$

General formula of the Laplace operator in orthogonal coordinates: In three-dimensional space, one of the fundamental properties of a coordinate system is orthogonality. If the coordinate axes are mutually perpendicular, such a coordinate system is called an orthogonal coordinate system.

In orthogonal coordinates, a point is defined by the coordinates (q_1, q_2, q_3) .

In these coordinates, the radius vector: $\vec{r} = \vec{r}(q_1, q_2, q_3)$ [39]

Derivatives with respect to coordinates: $\frac{\partial \vec{r}}{\partial q_1}, \frac{\partial \vec{r}}{\partial q_2}, \frac{\partial \vec{r}}{\partial q_3}$ It provides unit vectors in the coordinate directions. In an orthogonal system, these vectors are mutually orthogonal.

$$\frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j} = 0, \quad i \neq j \quad [40]$$

At each coordinate, the length scale in the direction is determined by the Lamé coefficients:

$$h_1 = \left| \frac{\partial \vec{r}}{\partial q_1} \right| \quad h_2 = \left| \frac{\partial \vec{r}}{\partial q_2} \right| \quad h_3 = \left| \frac{\partial \vec{r}}{\partial q_3} \right| \quad [41]$$

We write the differential form of the radius vector as follows:

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3 \quad [42]$$

Therefore, the unit direction vectors are:

$$\vec{e}_1 = \frac{1}{h_1} \frac{\partial \vec{r}}{\partial q_1} \quad \vec{e}_2 = \frac{1}{h_2} \frac{\partial \vec{r}}{\partial q_2} \quad \vec{e}_3 = \frac{1}{h_3} \frac{\partial \vec{r}}{\partial q_3}$$

So: $\frac{\partial \vec{r}}{\partial q_1} = h_1 \vec{e}_1 \quad \frac{\partial \vec{r}}{\partial q_2} = h_2 \vec{e}_2 \quad \frac{\partial \vec{r}}{\partial q_3} = h_3 \vec{e}_3$

If we substitute these expressions (43) into equation (42), we obtain the following form.

$$d\vec{r} = h_1 \vec{e}_1 dq_1 + h_2 \vec{e}_2 dq_2 + h_3 \vec{e}_3 dq_3 \quad [44]$$

The length element is determined by the following formula

$$ds = |d\vec{r}| \quad \text{bilan aniqlanadi}$$

Now we lift it onto the square.

$$ds^2 = d\vec{r} \cdot d\vec{r} \quad [45]$$

In orthogonal coordinates, since the axes are mutually orthogonal, the scalar product of unit vectors is zero, so only the squares of the unit vectors sum to one, yielding the following equality.

$$\vec{e}_1 \cdot \vec{e}_2 = 0 \quad \vec{e}_1 \cdot \vec{e}_3 = 0 \quad \vec{e}_2 \cdot \vec{e}_3 = 0 \quad \vec{e}_1 \cdot \vec{e}_1 = 1 \quad \vec{e}_2 \cdot \vec{e}_2 = 1 \quad \vec{e}_3 \cdot \vec{e}_3 = 1 \quad [46]$$

Using the equations in (45), if we substitute them into (45) and perform the calculation, we obtain the following result.

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 \quad [47]$$

$$ds = \sqrt{h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2}$$

$h_1 dq_1 - q_1$ Small length in the direction

$h_2 dq_2 - q_2$ Small length in the direction

$h_3 dq_3 - q_3$ Small length in the direction

Small length in the direction. For example, in Cartesian coordinates:

$$q_1 = x, \quad q_2 = y, \quad q_3 = z$$

$$h_1 = h_2 = h_3 = 1$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

At the pole coordinates:

$$\begin{aligned} q_1 &= r, & q_2 &= \varphi \\ h_1 &= 1, & h_2 &= r \\ ds^2 &= dr^2 + r^2 d\varphi^2 \end{aligned}$$

In a sphere, it is as follows:

$$\begin{aligned} q_1 &= r, & q_2 &= \theta, & q_3 &= \varphi \\ h_1 &= 1, & h_2 &= r, & h_3 &= r \sin \theta \\ ds^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \end{aligned} \quad [47]$$

Gradient operator in orthogonal coordinates:

$$\nabla u = \frac{1}{h_1} \frac{\partial u}{\partial q_1} \vec{e}_1 + \frac{1}{h_2} \frac{\partial u}{\partial q_2} \vec{e}_2 + \frac{1}{h_3} \frac{\partial u}{\partial q_3} \vec{e}_3 \quad [48]$$

will appear in the form

The Laplace operator is defined by the divergence of the gradient:

$$\Delta u = \nabla \cdot (\nabla u) \quad [49]$$

Divergence formula in orthogonal coordinates

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (h_2 h_3 A_1) + \frac{\partial}{\partial q_2} (h_1 h_3 A_2) + \frac{\partial}{\partial q_3} (h_1 h_2 A_3) \right] \quad [50]$$

then

$$\vec{A} = \Delta u$$

Let us assume. In that case:

$$A_1 = \frac{1}{h_1} \frac{\partial u}{\partial q_1} \quad A_2 = \frac{1}{h_2} \frac{\partial u}{\partial q_2} \quad A_3 = \frac{1}{h_3} \frac{\partial u}{\partial q_3} \quad [51]$$

(51) expressions, we now substitute them into the equation (50) and obtain the following result.

$$\Delta u = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial u}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial u}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial u}{\partial q_3} \right) \right] \quad [52]$$

We apply the Laplace operator to the length element of the spherical coordinate system introduced above. To do this, we perform the following steps.

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin \theta$$

Using these expressions, we find the products in expression (52) one after another. We multiply them by each other.

$$h_1 h_2 h_3 = r^2 \sin \theta$$

$$h_1 h_2 h_3 = 1 \cdot r \cdot r \sin \theta$$

Let's calculate the first limit:

$$\begin{aligned} \frac{h_2 h_3}{h_1} &= \frac{r \cdot r \sin \theta}{1} \\ \frac{h_2 h_3}{h_1} &= r^2 \sin \theta \end{aligned}$$

$$\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial u}{\partial r} \right)$$

Let's calculate the second limit:

$$\frac{h_1 h_3}{h_2} = \frac{1 \cdot r \sin \theta}{r}$$

$$\begin{aligned} \frac{h_1 h_3}{h_2} &= \sin \theta \\ \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) \end{aligned}$$

Let's calculate the third limit:

$$\frac{h_1 h_2}{h_3} = \frac{1 \cdot r}{r \sin \theta}$$

$$\begin{aligned} \frac{h_1 h_2}{h_3} &= \frac{1}{\sin \theta} \\ \frac{\partial}{\partial \varphi} \left(\frac{1}{\sin \theta} \frac{\partial u}{\partial \varphi} \right) \end{aligned}$$

Now we bring all the expressions we've found to the above [52] expression.

$$\Delta u = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{\sin \theta} \frac{\partial u}{\partial \varphi} \right) \right]$$

Now we simplify the expression step by step, starting by expanding the first term.

$$\begin{aligned} \frac{\partial}{\partial r} (r^2 \sin \theta u_r) &= \sin \theta \frac{\partial}{\partial r} (r^2 u_r) \\ \frac{\partial}{\partial r} (r^2 u_r) &= 2r u_r + r^2 u_{rr} \\ \sin \theta \frac{\partial}{\partial r} (r^2 u_r) &= \sin \theta (2r u_r + r^2 u_{rr}) \\ \frac{1}{r^2 \sin \theta} \sin \theta (2r u_r + r^2 u_{rr}) &= \frac{2r u_r + r^2 u_{rr}}{r^2} = u_{rr} + \frac{2}{r} u_r \end{aligned}$$

We will open the second boundary.

$$\frac{\partial}{\partial \theta} (\sin \theta u_\theta) = \cos \theta u_\theta + \sin \theta u_{\theta\theta}$$

$$\frac{1}{r^2 \sin \theta} (\cos \theta u_\theta + \sin \theta u_{\theta\theta}) = \frac{\cos \theta}{r^2 \sin \theta} u_\theta + \frac{1}{r^2} u_{\theta\theta} = \frac{1}{r^2} (u_{\theta\theta} + \cot \theta u_\theta)$$

We will open the 3rd border.

$$\frac{\partial}{\partial \varphi} \left(\frac{1}{\sin \theta} u_\varphi \right) = \frac{1}{\sin \theta} u_{\varphi\varphi}$$

$$\frac{1}{r^2 \sin \theta} \cdot \frac{1}{\sin \theta} u_{\varphi\varphi} = \frac{1}{r^2 \sin^2 \theta} u_{\varphi\varphi}$$

Now, by adding together all the limits we have expanded, we write the expression as follows:

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left(u_{\theta\theta} + \cot \theta u_\theta + \frac{1}{\sin^2 \theta} u_{\varphi\varphi} \right) \quad [53]$$

[53] The equation will be the Laplace operator in spherical coordinates.

4. Conclusion

The study theoretically derived the forms of the Laplace equation in polar and spherical coordinate systems and analyzed the steps of their derivation. It was shown that using polar and spherical coordinates allows for the simplification of symmetric mathematical physics problems and their efficient solution. The obtained results serve as an important theoretical foundation for the study of Laplace and Poisson equations.

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